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Asymptotic expansions for bivariate normal extremes

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ABSTRACT

Nadarajah (2015) derived a complete asymptotic expansion for normal extremes. Here, we extend the expansion for bivariate normal extremes.

1. Introduction

Let $\Phi(\cdot)$ denote the cumulative distribution function (cdf) of a standard normal random variable. It is well known that $\Phi(\cdot)$ belongs to the max domain of attraction of the Gumbel extreme value distribution, i.e.,

$$\Phi^n\left(u_n(x)\right) \to \exp\left(-\exp(-x)\right) \tag{1}$$

as $n \to \infty$ and for any $-\infty < x < +\infty$, where

$$u_n(x) = a_n x + b_n \tag{2}$$

and

$$a_n = (2\log n)^{-1/2}, \qquad b_n = a_n^{-1} - \frac{a_n c_n}{2} \tag{3}$$

for $n \geq 1$, where $c_n = \log\log n + \log(4\pi)$.

Nadarajah (2015) provided a complete asymptotic expansion for (1). In particular, an expansion was provided for

$$\Phi^n\left(u_n(x)\right) - \sum_{m=0}^{n} \frac{(-1)^m \exp(-mx)}{m!}$$

as $n \to \infty$, where u_n, a_n and b_n are given by (2)–(3).

The aim of this note is to extend Nadarajah (2015)'s work for the bivariate normal distribution given by the joint cdf

$$F(x,y) = \frac{1}{2\pi\sqrt{1-\rho^2}} \int_{-\infty}^{y}\int_{-\infty}^{x} \exp\left(-\frac{u^2+v^2-2\rho uv}{2\left(1-\rho^2\right)}\right) du dv$$

for $-\infty < x < +\infty$, $-\infty < y < +\infty$ and $-1 < \rho < 1$. It is well known that

$$F^n\left(u_n(x), u_n(y)\right) \to \exp\left(-\exp(-x) - \exp(-y)\right) \tag{4}$$

as $n \to \infty$, where u_n, a_n and b_n are given by (2)–(3).

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To the best of our knowledge, we are aware of no studies on convergence aspects of (4). However, there have been studies on convergence aspects of the extremes of other multivariate distributions and multivariate processes, see Hashorva and Ji (2014a,b, 2015, 2016), Hashorva and Kortschak (2014), Hashorva (2015), Hashorva et al. (2015, 2016), Hashorva and Li (2015) and Hashorva and Ling (2016).

Asymptotic expansions for (4) could have both practical and theoretical appeal given the wide applicability of the bivariate normal distribution. In a practical sense, they could lead to better approximations for the limit $\exp(-\exp(-x) - \exp(-y))$. Theoretically, such expansions can be used to derive expansions for the corresponding joint probability density function (pdf), moments, cumulants, quantiles, etc.

Complete asymptotic expansions for (4) are derived in Section 2. Technical lemmas for their proof are given and proved in Section 3. The expansions involve Bell polynomials. In-built routines for Bell polynomials are available in most computer algebra packages. For example, see BellY in Mathematica.

The following notation is used throughout this note: $\mathbf{0}_k$ denotes a $k \times 1$ vector of zeros; $\mathbf{1}_k$ denotes a $k \times 1$ vector of ones; ∞_k denotes a $k \times 1$ vector of infinities; $|\mathbf{a}| = a_1 + a_2 + \cdots + a_k$ for a $k \times 1$ vector $\mathbf{a} = (a_1, a_2, \ldots, a_k)'$; $\sum_{\mathbf{m}=\mathbf{0}_k}^{\infty_k}$ denotes the k fold summation $\sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} \cdots \sum_{m_k=0}^{\infty}$; $(a)_k = a(a+1)\cdots(a+k-1)$ denotes the ascending factorial; $B_{r,k}(\mathbf{x})$ denotes the *partial exponential Bell polynomial* defined by

$$\left(\sum_{r=1}^{\infty} x_r t^r / r! \right)^k \Big/ k! = \sum_{r=k}^{\infty} B_{r,k}(\mathbf{x}) t^r / r!$$

for $\mathbf{x} = (x_1, x_2, \ldots)$. This polynomial is tabled on page 307 of Comtet (1974) for $r \leq 12$.

Our results in Sections 2 and 3 can in principle be extended to any other joint cdf. We have illustrated our results for the bivariate normal distribution because of its universality.

2. Main results

Theorem 1 derives an asymptotic expansion for $F^n\left(u_n(x), u_n(y)\right)$. Theorem 2 uses Theorem 1 to prove (4). Theorems 3 and 4 derive complete asymptotic expansions for

$$\Phi^n\left(u_n(x)\right) \Phi^n\left(u_n(y)\right) - \sum_{m_1=0}^{n} \sum_{m_2=0}^{n} \frac{(-1)^{m_1+m_2}}{m_1! m_2!} \exp\left(-m_1 x - m_2 y\right)$$

and

$$F^n\left(u_n(x), u_n(y)\right) - \sum_{m_1=0}^{n} \sum_{m_2=0}^{n} \frac{(-1)^{m_1+m_2}}{m_1! m_2!} \exp\left(-m_1 x - m_2 y\right),$$

respectively, where u_n, a_n and b_n are given by (2)–(3). Proofs are given for Theorems 1 and 2. These proofs involve the use of Lemmas 1–9 in Section 3. Theorems 3 and 4 are straightforward consequences of Theorems 1 and 2, so their proofs are not given.

Theorem 1. *For u_n, a_n and b_n given by (2)–(3),*

$$\begin{aligned} F^n\left(u_n(x), u_n(y)\right) = & \sum_{m_1,p_1,q_1,i,\mathbf{i}}^{(1)} \sum_{m_2,p_2,q_2,j,\mathbf{j}}^{(1)} \sum_{k=0}^{n} \binom{n}{k} \left(\frac{\pi}{2}\right)^k \sum_{\mathbf{m}_1,q_3,p_3,\ell,\boldsymbol{\ell}}^{(2)} \sum_{\mathbf{m}_2,q_4,p_4,r,\mathbf{r}}^{(2)} \sum_{\mathbf{p},\mathbf{q},\mathbf{s},u,v}^{(3)} \\ & \cdot \lambda\left(m_1, p_1, q_1, i, \mathbf{i}, m_2, p_2, q_2, j, \mathbf{j}, \mathbf{m}_1, p_3, q_3, \ell, \boldsymbol{\ell}, \mathbf{m}_2, p_4, q_4, r, \mathbf{r}, \mathbf{p}, \mathbf{q}, \mathbf{s}, u, v, x, y\right) \\ & \cdot c_n^{2i_2+i_3+2j_2+j_3+2\ell_2+\ell_3+2r_2+r_3} \\ & \cdot n^{-m_1-m_2-|\mathbf{m}_1|-|\mathbf{m}_2|-2k} \\ & \cdot a_n^{2|\mathbf{i}|-m_1+i+2|\mathbf{j}|-m_2+j+\ell+2|\boldsymbol{\ell}|-|\mathbf{m}_1|+r+2|\mathbf{r}|-|\mathbf{m}_2|-2k-u-v} \\ & \cdot b_n^{-m_1-2q_1-i-m_2-2q_2-j-|\mathbf{m}_1|-2p_3-\ell-|\mathbf{m}_2|-2p_4-r-2|\mathbf{p}|-2|\mathbf{q}|-s_1-s_2+u+v} \end{aligned} \tag{5}$$

as $n \to \infty$, where $\lambda = \gamma\left(m_1, p_1, q_1, i, \mathbf{i}, x\right) \gamma\left(m_2, p_2, q_2, j, \mathbf{j}, y\right) \delta\left(\mathbf{m}_1, p_3, q_3, \ell, \boldsymbol{\ell}, x\right) \delta\left(\mathbf{m}_2, p_4, q_4, r, \mathbf{r}, y\right) \epsilon(\mathbf{p}, \mathbf{q}, \mathbf{s}, u, v, x, y)$ and $\gamma\left(m_1, p_1, q_1, i, \mathbf{i}, x\right)$, $\gamma\left(m_2, p_2, q_2, j, \mathbf{j}, y\right)$, $\delta\left(\mathbf{m}_1, p_3, q_3, \ell, \boldsymbol{\ell}, x\right)$, $\delta\left(\mathbf{m}_2, p_4, q_4, r, \mathbf{r}, y\right)$, $\epsilon\left(\mathbf{p}, \mathbf{q}, \mathbf{s}, u, v, x, y\right)$, $\sum_{m_1,p_1,q_1,i,\mathbf{i}}^{(1)}$, $\sum_{m_2,p_2,q_2,j,\mathbf{j}}^{(1)}$, $\sum_{\mathbf{m}_1,q_3,p_3,\ell,\boldsymbol{\ell}}^{(2)}$, $\sum_{\mathbf{m}_2,q_4,p_4,r,\mathbf{r}}^{(2)}$, $\sum_{\mathbf{p},\mathbf{q},\mathbf{s},u,v}^{(3)}$ are as defined in Lemmas 1–9.

Proof. By Balakrishnan and Lai (2009, page 493, Equation (11.30)),

$$F(x, y) = \Phi(x)\Phi(y) + \phi(x)\phi(y) \sum_{j=1}^{\infty} \frac{\rho^j}{j} H_{j-1}(x) H_{j-1}(y),$$