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Moment convergence of first-passage times in renewal theory

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ABSTRACT

Let $\xi_1, \xi_2, \ldots$ be independent copies of a positive random variable ξ, $S_0 = 0$, and $S_k = \xi_1 + \ldots + \xi_k$, $k \in \mathbb{N}$. Define $N(t) = \inf\{k \in \mathbb{N} : S_k > t\}$ for $t \geq 0$. The process $(N(t))_{t\geq 0}$ is the first-passage time process associated with $(S_k)_{k\geq 0}$. It is known that if the law of ξ belongs to the domain of attraction of a stable law or $\mathbb{P}(\xi > t)$ varies slowly at ∞, then $N(t)$, suitably shifted and scaled, converges in distribution as $t \to \infty$ to a random variable W with a stable law or a Mittag-Leffler law. We investigate whether there is convergence of the power and exponential moments to the corresponding moments of W. Further, the analogous problem for first-passage times of subordinators is considered.

1. Introduction and results

Setup. Let $\xi_1, \xi_2, \ldots$ be independent copies of a positive random variable ξ. We set $\mu := \mathbb{E}[\xi] \in (0, \infty]$, and then $\sigma^2 := \mathbb{V}\mathrm{ar}[\xi]$ whenever μ is finite. Throughout the paper, we assume that the law of ξ is non-degenerate, that is, $\mathbb{P}(\xi = c) < 1$ for all $c > 0$. Define

$$S_0 := 0, \qquad S_k := \xi_1 + \cdots + \xi_k, \quad k \in \mathbb{N},$$

and

$$N(t) := \#\{k \in \mathbb{N}_0 : S_k \leq t\} = \inf\{k \in \mathbb{N} : S_k > t\}, \quad t \geq 0.$$

The stochastic process $(N(t))_{t\geq 0}$ is called *first-passage time process* associated with $(S_k)_{k\geq 0}$. The term 'renewal counting process' is also used.

Objective. It is known (see, for instance, Gnedin et al. (2009, Proposition A.1)) that if the law of ξ is in the domain of attraction of a stable law or $\mathbb{P}(\xi > t)$ varies slowly at ∞, then

$$\frac{N(t) - b(t)}{a(t)} \stackrel{\mathrm{d}}{\to} W \quad \text{as } t \to \infty \tag{1.1}$$

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where "$\xrightarrow{\mathrm{d}}$" denotes convergence in distribution, W is a non-degenerate random variable, and $b(t) \in \mathbb{R}$, $a(t) > 0$ are suitable shifting and scaling functions, respectively.

The purpose of this note is to answer the question: when does (1.1) imply convergence of the corresponding power and exponential moments, finite or infinite? The motivation for writing a short note on this problem comes from the fact that the moment convergence of first-passage time processes repeatedly turned out to be an important technical step in other works on processes bearing some regenerative or renewal structure. For instance, Theorems 1.1 and 1.4 are essential ingredients in our work on the finite-dimensional convergence of shot noise processes (Iksanov et al., 2014). Theorem 1.5 is used to prove convergence of shot noise processes to fractionally integrated inverse stable subordinators (Iksanov et al., 2016). Corollary 1.6 is used in the proof of Theorem 3.3 in Gnedin and Iksanov (2012). Consequently, we found it useful to have one paper which contains the complete results on convergence of power and exponential moments for renewal counting processes.

Before we state our results we briefly recall the different regimes in which (1.1) holds.

Domains of attraction. The law of a random variable ξ is in the domain of attraction of an α-stable law, $\alpha \in (0, 2]$ or $\mathbb{P}\{\xi > t\}$ varies slowly at ∞ if one of the following alternatives prevails[1]:

(A1) $\mu < \infty$ and $\sigma^2 := \mathbb{V}\mathrm{ar}[\xi] < \infty$;
(A2) $\mu < \infty$ but $\sigma^2 = \infty$ and $\ell_2(t) := \mathbb{E}[\xi^2 \mathbb{1}_{\{\xi \le t\}}]$ is slowly varying at ∞;
(A3) $\mathbb{P}(\xi > t) = t^{-\alpha}\ell(t)$ for some $\alpha \in (1, 2)$ and a function ℓ slowly varying at ∞;
(A4) $\mathbb{P}(\xi > t) = t^{-\alpha}\ell(t)$ for some $\alpha \in [0, 1)$ and a function ℓ slowly varying at ∞.

We refer to Ibragimov and Linnik (1971, Section 2.6) for details. The convergence of the first-passage time process in (1.1) can now be described more precisely:

(N1) if (A1) holds, then $b(t) = t/\mu$, $a(t) = \sigma\mu^{-3/2}c(t)$, $c(t) = \sqrt{t}$, and W is a standard normal random variable;
(N2) if (A2) holds, then $b(t) = t/\mu$, $a(t) = \mu^{-3/2}c(t)$ where $c(t)$ is a positive function satisfying $\lim_{t\to\infty} t\ell_2(c(t))c(t)^{-2} = 1$, and W is a standard normal random variable;
(N3) if (A3) holds, then $b(t) = t/\mu$, $a(t) = \mu^{-(1+\alpha)/\alpha}c(t)$ where $c(t)$ is a positive function such that $\lim_{t\to\infty} t\ell(c(t))c(t)^{-\alpha} = 1$, and W is a random variable with characteristic function given by[2]

$$\psi(\lambda) = \exp\big\{-|\lambda|^\alpha \Gamma(1-\alpha)(\cos(\pi\alpha/2) + \mathrm{i}\sin(\pi\alpha/2)\mathrm{sgn}(\lambda))\big\}, \quad \lambda \in \mathbb{R} \tag{1.2}$$

where $\Gamma(\cdot)$ denotes Euler's gamma function;
(N4) if (A4) holds, then $b(t) = 0$, $a(t) = 1/\mathbb{P}(\xi > t)$, and W has a Mittag-Leffler distribution with parameter α (exponential with mean 1 if $\alpha = 0$), that is, W has moment generating function

$$\mathbb{E}[e^{\theta W}] = E_\alpha\Big(\frac{\theta}{\Gamma(1-\alpha)}\Big) < \infty, \quad \theta \in \mathbb{R}$$

where here and throughout the paper, E_α is the Mittag-Leffler function with parameter α given by $E_\alpha(z) := \sum_{k\ge0} \frac{z^k}{\Gamma(k\alpha+1)}$ for $z \in \mathbb{R}$.

Main results for random walks. In what follows we use the notation x_- and x_+ for the negative and positive part of a real number x:

$$x_- := -\min\{x, 0\} \quad \text{and} \quad x_+ := \max\{x, 0\}.$$

Theorem 1.1. *Suppose that either* (A1) *or* (A2) *holds, i.e.,* $\mu < \infty$ *and either* $\sigma^2 < \infty$ *or* $\sigma^2 = \infty$ *and* $\ell_2(t) := \mathbb{E}[\xi^2\mathbb{1}_{\{\xi\le t\}}]$ *is slowly varying at* ∞*. Then*

$$\lim_{t\to\infty} \mathbb{E}\Big[\exp\Big(\theta\frac{N(t) - t/\mu}{a(t)}\Big)\Big] = \mathbb{E}[e^{\theta W}] = e^{\frac{\theta^2}{2}}, \quad \text{for every } \theta \ge 0 \tag{1.3}$$

where W *is standard normal,* $a(t) = \sqrt{\sigma^2\mu^{-3}t}$ *in the case* (A1) *and* $a(t) = \mu^{-3/2}c(t)$ *for a positive function* $c(t)$ *satisfying* $\lim_{t\to\infty} t\ell_2(c(t))c(t)^{-2} = 1$ *in the case* (A2)*. In particular,*

$$\lim_{t\to\infty} \mathbb{E}\Big[\Big(\frac{N(t) - t/\mu}{a(t)}\Big)_+^p\Big] = \mathbb{E}[W_+^p] = \frac{2^{p/2-1}\Gamma\big(\frac{p+1}{2}\big)}{\sqrt{\pi}} \quad \text{for every } p > 0. \tag{1.4}$$

[1] Here, we do not treat the case where $\mathbb{P}(\xi > t)$ is regularly varying of index -1 at ∞ as it appears less frequently in applications and requires cumbersome calculations that would impair the character of this paper as a brief note.
[2] For $\alpha \in (1, 2)$, $\Gamma(1-\alpha)$ is understood as $-\Gamma(2-\alpha)/(\alpha-1)$.