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Moment for the inverse Riesz distributions

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ABSTRACT

The Riesz distributions dealing with positive definite symmetric matrices are usually used to introduce the class of the inverse Riesz distributions. The latter represents the natural extension of the class of the inverse Wishart. In this paper, we first present a sufficient condition allowing the existence of the expectation of the inverse Riesz distribution. Then, we compute it explicitly. For this purpose, we basically use the Cholesky decomposition as well as an important relation satisfied by the first derivative of continuous Riesz distribution's density. The importance of this first moment consists in the fact that it can be used to estimate the shape parameter through the method of moments.

1. Introduction

The Riesz distribution (also called the Bellman gamma distribution) on the space of real symmetric matrices equipped with the scalar product $\langle x, y\rangle = \mathrm{tr}(xy)$, where $\mathrm{tr}(xy)$ is the trace of matrices' ordinary product, was introduced in 2001 by Hassairi and Lajmi (2001). It was based on the notion of generalized power and was dependent on two parameters s and σ. The first one is the scale parameter s which belongs to a subset of $\mathbb{R}^r$ and for which we have a convolution semi-group. The second one is the shape parameter σ which is an element of the cone Ω of $r \times r$ real positive definite symmetric matrices. More precisely, the Riesz random matrix on the cone Ω, depending on $s \in \prod_{i=1}^{r}](i-1)/2, +\infty[$ and $\sigma \in \Omega$, has the following continuous probability density function

$$R(s, \sigma)(x) = \frac{\exp(-\langle \sigma, x\rangle)\Delta_{s-\frac{n}{r}}(x)}{\Gamma_\Omega(s)\Delta_s(\sigma^{-1})}\mathbf{1}_\Omega(x), \tag{1.1}$$

where $n = r(r+1)/2$ is the dimension of the space E of symmetric $r \times r$ matrices, Γ_Ω is the multivariate gamma function

$$\Gamma_\Omega(s) = (2\pi)^{\frac{n-r}{2}} \prod_{i=1}^{r} \Gamma\left(s_i - (i-1)/2\right) \tag{1.2}$$

and $\Delta_s(x)$ is the generalized power of x in the cone Ω defined, for $s \in \mathbb{R}^r$, by

$$\Delta_s(x) = \Delta_1(x)^{s_1-s_2}\Delta_2(x)^{s_2-s_3}\dots\Delta_r(x)^{s_r}, \tag{1.3}$$

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where, for any matrix $x = (x_{ij})_{1\le i,j\le r}$ and $1 \le k \le r$, $\Delta_k(x)$ is the determinant of the $k \times k$ sub-matrix $P_k(x) = (x_{ij})_{1\le i,j\le k}$ of x. The class of the Riesz distributions on the cone of positive matrices of rank r represents a natural extension of the class of the Wishart distributions on the space E of symmetric matrices. In fact, if for all $i \in \{1, \dots, r\}$, $s_i = p \in \mathbb{R}$, then $\Delta_s(x) = (\det x)^p$ and the Riesz distribution $R(s, \sigma)$ when $s_1 = \cdots = s_r = p > (r-1)/2$, reduces to the Wishart distribution

$$W(p,\sigma)(dx) = \frac{e^{-<\sigma,x>}\det(x)^{p-\frac{n}{r}}}{\Gamma_\Omega(p)\det(\sigma^{-p})}\mathbf{1}_\Omega(x)(dx)$$

which constitutes a fundamental distribution in multivariate statistical analysis. It has attracted, for a long time, many researchers. Nevertheless, many results have been obtained only recently (see for instance Andersson and Klein, 2010, Letac, 1989 and Massam and Wesołowski, 2004). From a theoretical point of view, several interesting results concerning the Riesz distribution were obtained (see for instance Díaz-Garcia, 2013, Hassairi and Lajmi, 2001, Hassairi and Louati, 2009, 2013 and Louati, 2013). In practice, it arises as a distribution of the empirical Normal covariance matrix for samples with monotonous missing data (see Veleva, 2009).

If X is a Riesz random matrix, then the distribution of X^{-1} is called the inverse Riesz distribution (see Tounsi and Zine, 2012) which can be regarded as a generalization of the inverse Wishart distribution (see Haff, 1982 or von Rosen, 1988). It verifies the property of set-positivity given by the covariance matrices. For this reason, the inverse Riesz distribution can be applied for the estimation of a covariance matrix of a multivariate normal distribution.

The aim of this paper is to calculate the expectation of the inverse Riesz distribution. For this reason, we first give a sufficient condition for its existence. In order to do this, we may use Weyl's integration formula which is fundamental in representation theory and also in other areas like random matrix theory. Then, we compute it explicitly. This is performed by the simultaneous use of the Cholesky decomposition and an important relation satisfied by the first derivative of the probability density of the Riesz distribution given by (1.1).

The paper is organized as follows: Section 2 presents some definitions and gives some preliminary results dealing with the Riesz distributions. In Section 3, we start by studying the existence of the first moment of the inverse Riesz distribution by using a suitable formula allowing this, namely Weyl's integration formula. Then, we calculate it explicitly by using basically the Cholesky decomposition.

2. Riesz distributions

To clarify the results of this paper, we first recall some notations and review some characteristic properties concerning the Riesz distributions on the Euclidean space E of $r \times r$ real symmetric matrices which is equipped with the scalar product $\langle x, y\rangle = \mathrm{tr}(xy)$, where xy is an ordinary product of the matrices. Our notations are the ones used in the book of Faraut and Korányi (1994) and the paper of Hassairi and Lajmi (2001).

Let Ω be the cone of positive definite elements of E and let $\mathcal{M}(E)$ be the set of σ-finite positive measures on E, which are not concentrated on an affine hyperplane of E such that $\Theta(\mu)$ the interior of the domain of its Laplace transform

$$L_\mu(\theta) = \int_E \exp(\langle\theta, x\rangle)\mu(dx) < \infty$$

is not empty. For each $\mu \in \mathcal{M}(E)$ and $\theta \in \Theta(\mu)$, the cumulant function of the measure μ is defined on $\Theta(\mu)$ by

$$k_\mu(\theta) = \log\left(L_\mu(\theta)\right),$$

(see Kokonendji and Khoudar, 2006). According to Gindikin, 1964, his result leads to the fact that there exists a positive measure R_s such that, for all $\theta \in -\Omega$,

$$L_{R_s}(\theta) = \int_E e^{\langle\theta,x\rangle}R_s(dx) = \Delta_s(-\theta^{-1})$$

if, and only if, s is in a subset Ξ of $\mathbb{R}^r$ containing $\prod_{i=1}^r](i-1)/2, +\infty[$ (for a more precise statement in this connection, refer to Faraut and Korányi, 1994, p. 124), where Δ_s is defined in (1.3). The measure R_s is called the Riesz measure with a parameter s and if $s \in \prod_{i=1}^r](i-1)/2, +\infty[$,

$$R_s(dx) = \frac{\Delta_{s-\frac{n}{r}}(x)}{\Gamma_\Omega(s)}\mathbf{1}_\Omega(x)dx,$$

where Γ_Ω was already defined in (1.2). The following distribution

$$R(s,\sigma)(dx) = \frac{\exp(-\langle\sigma,x\rangle)}{\Delta_s(\sigma^{-1})}R_s(dx) = \frac{\exp(-\langle\sigma,x\rangle)\Delta_{s-\frac{n}{r}}(x)}{\Gamma_\Omega(s)\Delta_s(\sigma^{-1})}\mathbf{1}_\Omega(dx)$$

is called the Riesz distribution with a scale parameter $s \in \prod_{i=1}^r](i-1)/2, +\infty[$ and a shape parameter $\sigma \in \Omega$. It is characterized by its Laplace transform equal to

$$L_{R(s,\sigma)}(\theta) = \frac{\Delta_s(\sigma-\theta)^{-1}}{\Delta_s(\sigma^{-1})} \quad \text{for all } \theta \in \sigma - \Omega. \tag{2.4}$$