Statistics and Probability Letters 102 (2015) 38–41

Contents lists available at ScienceDirect

Statistics and Probability Letters

journal homepage: www.elsevier.com/locate/stapro

Diffusion hitting times and the bell-shape

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ARTICLE INFO

Article history:
Received 13 February 2015
Received in revised form 24 March 2015
Accepted 24 March 2015
Available online 7 April 2015

MSC:
60E05
60J60

Keywords:
Bell-shape
Exponential mixture
Generalized diffusion
Hitting time
Speed measure

ABSTRACT

Consider a generalized diffusion on $\mathbb{R}$ with speed measure m, in the natural scale. It is known that the conditional hitting times have a unimodal density function. We show that these hitting densities are bell-shaped if and only if m has infinitely many points of increase between the starting point and the hit point. This result can be viewed as a visual corollary to Yamazato's general factorization for diffusion hitting times.

1. Introduction and statement of the result

This paper deals with a certain distributional property of hitting times for generalized diffusions. We use the standard notation for the latter processes, as described e.g. in Chapter V of Rogers and Williams (1987). Let m be a string, that is a right continuous non-decreasing function from $[-\infty, +\infty]$ to $[-\infty, +\infty]$, with $m(-\infty) = -\infty, m(+\infty) = +\infty$ and $m(0-) = 0$. Set

$$r_1 = \sup\{x < 0,\ m(x) = -\infty\}, \qquad r_2 = \inf\{x > 0,\ m(x) = +\infty\},$$

and define the positive measure $m(dx)$ on $[-\infty, +\infty]$ by

$$m(dx) = dm(x) \quad \text{on } (r_1, r_2), \qquad m([r_1, r_2]^c) = 0 \quad \text{and} \quad m(\{r_1\}) = m(\{r_2\}) = +\infty.$$

Let $\{B_t,\ t \geq 0\}$ be a linear Brownian motion and $\{L_t^x,\ t \geq 0,\ x \in \mathbb{R}\}$ be its local time. Introducing the additive functional

$$A_t = \int_{\mathbb{R}} L_t^x\, m(dx)$$

and its right-continuous inverse $\tau_t = \inf\{u > 0,\ A_u > t\}$, we define the process

$$X_t = B_{\tau_t}, \quad 0 \leq t < \zeta,$$

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http://dx.doi.org/10.1016/j.spl.2015.03.008
0167-7152/

with lifetime $\zeta = \inf\{t > 0, X_t = r_1 \text{ or } r_2\}$. Here and throughout, the notation $\inf \emptyset = +\infty$ is implicitly assumed. The process X is strongly Markovian with state space $E_m = (r_1, r_2) \cap \text{Supp}\, m$, and we set $\mathbb{P}_x$ for its law starting from $x \in E_m$. The measure m is called the speed measure of X. As is well-known, the above time-changed Brownian motion allows to construct all linear diffusions on an interval up to some monotonous, scale transformation—see again Chapter V. 7 in Rogers and Williams (1987). One can also choose a speed measure with discrete support, which leads to a class of continuous time Markov chains called gap diffusions in the literature—see Kotani and Watanabe (1982) for details and examples. The above generalized diffusion is chosen on the natural scale, which does not cause any loss of generality in our problem.

The hitting time of $y \in E_m$ is defined by

$$\tau_y = \inf\{t > 0;\ X_t = y\}.$$

This definition extends to $y = r_i$ for $i = 1, 2$ if we suppose $|r_i| < \infty$ and $r_i \in \bar{E}_m$, setting $\tau_{r_i} = \lim_{y \to r_i} \tau_y$. In this situation we adjoin r_i to E_m and denote by $\tilde{E}_m$ the extended state space. Let $x \in E_m$ and $y \neq x \in \tilde{E}_m$ be such that $\mathbb{P}_x[\tau_y < \infty] > 0$. Without loss of generality we will suppose $y > x$. This paper deals with the conditional hitting time distribution

$$\pi_{xy}(dt) = \frac{\mathbb{P}_x[\tau_y \in dt]}{\mathbb{P}_x[\tau_y < \infty]}.$$

It is well-known that the probability measure π_{xy} is absolutely continuous, and we denote by $f_{x,y}$ its density function on $(0, +\infty)$. It was shown in Theorem 1.2 of Rösler (1980) that this density is always unimodal, in other words that it has a unique local maximum.

In this paper, we are interested in the following refinement of unimodality for $f_{x,y}$. A smooth density function defined on a real interval is said to be *bell-shaped* if all its derivatives vanish at both ends of the interval and if its n-th derivative vanishes exactly n times, for all $n \geq 1$. Setting $n = 1$ shows that a bell-shaped density is strictly unimodal. For $n = 2$ the bell-shape property entails, as for the familiar bell curve, that there is one inflection point on each side of the mode and that the second derivative is successively positive, negative, and positive. The visual meaning of the bell-shape for $n = 3$ or 4 is less immediate and we refer to the introduction of Simon (2015) for details and references. Some standard density functions like the Cauchy, the Gaussian, the Gumbel or the Student are bell-shaped, as can be seen by a direct analysis. Showing this property for less explicit densities is however a more demanding task.

Consider now the generalized inverse Gaussian distribution, with density

$$c\, x^{\lambda-1}\, e^{-(\chi x^{-1} + \psi x)}$$

over $(0, +\infty)$, where c is the normalization constant and the variation domain of (λ, χ, ψ) is described in Barndorff-Nielsen et al. (1978, p. 49). If $\chi > 0$, the above density is bell-shaped, as can again be seen by a direct analysis, and it is also a hitting density for some SDE driven by Brownian motion—see Theorem 2.1. in Barndorff-Nielsen et al. (1978). If $\chi = 0$, the above density is not bell-shaped, and it is the hitting density of a generalized diffusion only if $\lambda \leq 1$—see Corollary 2 in Yamazato (1990). In this case it is also completely monotone, hence the first passage density to the nearest neighbour of some continuous time Markov chain with discrete state space—see Bondesson (1992, p. 147) and the references therein. If $\chi > 0$ the speed measure of the underlying diffusion has an everywhere positive density, whereas if $\chi = 0, \lambda \leq 1$, the involved speed measure is atomic. This example suggests a general relationship between the bell-shape property for the hitting times of a generalized diffusion and the support of its speed measure. This connection is illustrated by the following characterization.

Theorem. *The density $f_{x,y}$ is bell-shaped iff m has an infinite number of increase points on (x, y).*

This simple criterion applies to all SDE's driven by Brownian motion, and also to more singular processes like those of Examples 2.12 and 2.13 in Freedman (1971). There exists an immense literature on hitting times for real diffusions, however it seems that the above interesting distributional property has not been investigated as yet. Our method to prove the theorem relies on a general factorization by Yamazato (1990), and a total positivity argument which the first author had used in a previous paper (Simon, 2015) on stable densities. In this respect, it is worth recalling that diffusion hitting times are always infinitely divisible. The bell-shape property for all positive self-decomposable densities with infinite spectral function at zero is stated as an open problem in Simon (2015)—see Conjecture 1 therein.

2. Proof of the theorem

2.1. Proof of the if part

Translating the measure m if necessary, we may and will suppose $x = 0$ and $y > 0$. We appeal to the factorization obtained in Theorem 1 of Yamazato (1990), which reads

$$\pi_{0y} = \mu_1 * \mu_2,$$

where μ_1 has Laplace transform

$$\int_0^\infty e^{-\lambda x}\, \mu_1(dx) = \prod_i \left(\frac{a_i}{a_i + \lambda} \right)$$