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Stochastic comparisons of weighted sums of arrangement increasing random variables

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ABSTRACT

Assuming that the joint density of random variables $X_1, X_2, \dots, X_n$ is arrangement increasing (AI), we obtain some stochastic comparison results on weighted sums of X_i's under some additional conditions. An application to optimal capital allocation is also given.

1. Introduction

During the past few decades, linear combinations of random variables have been extensively studied in statistics, operations research, reliability theory, actuarial science and other fields. Most of the related work restricts to some specific distributions such as Exponential, Weibull, Gamma and Uniform, among others. Karlin and Rinott (1983) and Yu (2011) studied the stochastic properties of linear combinations of independent and identically distributed (i.i.d.) random variables without putting any distributional assumptions. Later on, Xu and Hu (2011, 2012), Pan et al. (2013) and Mao et al. (2013) weakened the i.i.d. assumption to independent, yet possibly non-identically distributed (i.ni.d), random variables. It should be noted that most of the related work assumes that the random variables are mutually independent.

Recently, some work has appeared on stochastic comparisons of dependent random variables. Xu and Hu (2012) discussed stochastic comparisons of comonotonic random variables with applications to capital allocations. You and Li (2014) focused on linear combinations of random variables with Archimedean dependence structure. Cai and Wei (2014) proposed

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several new notions of dependence to measure dependence between risks. They proved that characterizations of these notions are related to properties of arrangement increasing (AI) functions (to be defined in Section 2). Motivated by the importance of AI functions, we study the problem of stochastic comparisons of weighted sums of AI random variables in this paper.

We say $X_1, \ldots, X_n$ are AI random variables if their joint density $f(\boldsymbol{x})$ is an AI function. Ma (2000) proved the following result for AI random variables $X_1, \ldots, X_n$:

$$\mathbf{a} \succeq_{\mathrm{m}} \mathbf{b} \Longrightarrow \sum_{i=1}^{n} a_{(i)} X_i \geq_{\mathrm{icx}} \sum_{i=1}^{n} b_{(i)} X_i, \quad \forall\, \mathbf{a}, \mathbf{b} \in \Re^n, \tag{1.1}$$

where $a_{(1)} \leq a_{(2)} \leq \cdots \leq a_{(n)}$ is the increasing arrangement of the components of the vector $\mathbf{a} = (a_1, a_2, \ldots, a_n)$. The formal definitions of stochastic orders and majorization orders are given in Section 2.

Let $X_1, X_2, \ldots, X_n$ be independent random variables satisfying

$$X_1 \geq_{\mathrm{hr}} X_2 \geq_{\mathrm{hr}} \cdots \geq_{\mathrm{hr}} X_n,$$

and let $\phi(x, a)$ be a convex function which is increasing in x for each a. Mao et al. (2013) proved that

(i) if ϕ is submodular, then

$$\mathbf{a} \succeq_{\mathrm{m}} \mathbf{b} \Longrightarrow \sum_{i=1}^{n} \phi\left(X_i, a_{(i)}\right) \geq_{\mathrm{icx}} \sum_{i=1}^{n} \phi\left(X_i, b_{(i)}\right); \tag{1.2}$$

(ii) if ϕ is supermodular, then

$$\mathbf{a} \succeq_{\mathrm{m}} \mathbf{b} \Longrightarrow \sum_{i=1}^{n} \phi\left(X_i, a_{(n-i+1)}\right) \geq_{\mathrm{icx}} \sum_{i=1}^{n} \phi\left(X_i, b_{(n-i+1)}\right). \tag{1.3}$$

The function ϕ in (1.2) and (1.3) could be interpreted as some appropriate distance measures in actuarial science. For more details, please refer to Xu and Hu (2012).

In this paper we further study the problem of stochastic comparisons of linear combinations of AI random variables not only for increasing convex ordering, but also for the usual stochastic ordering. The rest of this paper is organized as follows. Some preliminaries are given in Section 2. The main results are presented in Section 3. An application to optimal capital allocation is discussed in Section 4.

2. Preliminaries

In this section, we give definitions of some stochastic orders, majorization orders and supermodular [submodular] functions. Throughout the paper, the terms 'increasing' and 'decreasing' are used to mean 'non-decreasing' and 'non-increasing', respectively.

Definition 2.1 (*Stochastic Orders*)**.** Let X and Y be two random variables with probability (mass) density functions f and g; and survival functions $\overline{F}$ and $\overline{G}$ respectively. We say that X is smaller than Y

(1) in the *usual stochastic order*, denoted by $X \leq_{\mathrm{st}} Y$, if $\overline{F}(t) \leq \overline{G}(t)$ for all t or, equivalently, if $\mathsf{E}[h(X)] \leq \mathsf{E}[h(Y)]$ for all increasing functions h;
(2) in the *hazard rate order*, denoted by $X \leq_{\mathrm{hr}} Y$, if $\overline{G}(t)/\overline{F}(t)$ is increasing in t for which the ratio is well defined;
(3) in the *likelihood ratio order*, denoted by $X \leq_{\mathrm{lr}} Y$, if $g(t)/f(t)$ is increasing in t for which the ratio is well defined;
(4) in the increasing convex order, denoted by $X \leq_{\mathrm{icx}} Y$, if $\mathsf{E}[h(X)] \leq \mathsf{E}[h(Y)]$ for all increasing convex functions h for which the expectations exist.

The relationships among these orders are shown in the following diagram (see Shaked and Shanthikumar, 2007 and Müller and Stoyan, 2002):

$$X \leq_{\mathrm{lr}} Y \Longrightarrow X \leq_{\mathrm{hr}} Y \Longrightarrow X \leq_{\mathrm{st}} Y \Longrightarrow X \leq_{\mathrm{icx}} Y.$$

Shanthikumar and Yao (1991) considered the problem of extending the above concepts to compare the components of dependent random variables. In this paper we will focus only on extension of likelihood ratio ordering to the case of dependent random variables. Let (X, Y) be a continuous bivariate random vector on $[0, \infty]^2$ with joint density (or mass) function $f(x, y)$.

Definition 2.2. For a bivariate random variable (X, Y), X is said to be smaller than Y according to *joint likelihood ordering*, denoted by $X \leq_{\ell r:j} Y$, if and only if

$$E[\Psi(X, Y)] \geq E[\Psi(Y, X)], \quad \Psi \in \mathcal{G}_{\ell r},$$

where

$$\mathcal{G}_{\ell r} : \{\Psi : \Psi(x, y) \geq \Psi(y, x),\ x \leq y\}.$$