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Consistency of the plug-in functional predictor of the Ornstein–Uhlenbeck process in Hilbert and Banach spaces

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ABSTRACT

New results on functional prediction of the Ornstein–Uhlenbeck process in an autoregressive Hilbert-valued and Banach-valued frameworks are derived. Specifically, consistency of the maximum likelihood estimator of the autocorrelation operator, and of the associated plug-in predictor is obtained in both frameworks.

1. Introduction

This paper derives new results in the context of linear processes in function spaces. An extensive literature has been developed in this context, in the last few decades (see, for example, Bosq, 2000; Ferraty and Vieu, 2006; Ramsay and Silverman, 2005, among others). In particular, the problem of functional prediction of linear processes in Hilbert and Banach spaces has been widely addressed. We refer to the reader to the papers by: Bensmain and Mourid (2001), Bosq (1996), Bosq (2002), Bosq (2004), Bosq and Blanke (2007), Dedecker and Merlevede (2003), Dehling and Sharipov (2005), Glendinning and Fleet (2007), Guillas (2000), Guillas (2001), Kargin and Onatski (2008), Labbas and Mourid (2003), Marion and Pumo (2004), Mas (2002), Mas (2004), Mas (2007), Mas and Menneteau (2003), Mas and Pumo (2007), Menneteau (2005), Mourid (2002), Mourid (2004), Mokhtari and Mourid (2002), Pumo (1998), Rachedi (2004), Rachedi (2005), Rachedi and Mourid (2003), Ruiz-Medina (2012), Turbillon et al. (2007), Turbillon et al. (2008), and the references therein. In the above-mentioned papers, different projection methodologies have been adopted in the derivation of the main asymptotic properties of the formulated functional parameter estimators and predictors. Particularly, Bosq (2000, 2007) apply Functional Principal Component Analysis; Antoniadis et al. (2006) and Antoniadis and Sapatinas (2003) consider wavelet bases;

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Laukaitis et al. (2009) propose wavelet estimation methods. Applications of these functional estimation results can be found in the papers by: Antoniadis and Sapatinas (2003), Damon and Guillas (2002), Hormann and Kokoszka (2011), Laukaitis (2008) and Ruiz-Medina and Salmerón (2009), among others.

We pay attention here to the problem of functional prediction of the Ornstein–Uhlenbeck (O.U.) process (see, for example, Uhlenbeck and Ornstein (1930), and Wang and Uhlenbeck (1945), for its introduction and properties). See also Doob (1942) for the classical definition of O.U. process from the Langevin (linear) stochastic differential equation. We can find in Kutoyants (2004) and Liptser and Shiraev (2001) an explicit expression of the maximum likelihood estimator (MLE) of the scale parameter θ, characterizing its covariance function. Its strong consistency is proved, for instance, in Kleptsyna and Le Breton (2002). We formulate here the O.U. process as an Autoregressive Hilbertian process of order one (ARH(1) process), and as an Autoregressive Banach-valued process of order one (ARB(1) process). Consistency of the MLE of θ is applied to prove consistency of the corresponding MLE of the autocorrelation operator of the O.U. process. We adopt the methodology applied in Bosq (1991), since our interest relies on forecasting the values of the O.U. process over an entire time interval. Specifically, considering the O.U. process $\{\xi_t\}_{t\in\mathbb{R}}$, on the basic probability space $(\Omega, \mathcal{A}, P)$, we can define

$$X_n(t) = \xi_{nh+t}, \quad 0 \leq t \leq h,\ n \in \mathbb{Z}, \tag{1}$$

satisfying

$$X_n(t) = \xi_{nh+t} = \int_{-\infty}^{nh+t} e^{-\theta(nh+t-s)} dW_s = \rho_\theta(X_{n-1})(t) + \varepsilon_n(t), \quad n \in \mathbb{Z}, \tag{2}$$

with

$$\rho_\theta(x)(t) = e^{-\theta t} x(h), \qquad \rho_\theta(X_{n-1})(t) = e^{-\theta t} \int_{-\infty}^{nh} e^{-\theta(nh-s)} dW_s, \qquad \varepsilon_n(t) = \int_{nh}^{nh+t} e^{-\theta(nh+t-s)} dW_s, \tag{3}$$

for $0 \leq t \leq h$. Thus, $X = (X_n,\ n \in \mathbb{Z})$ satisfies the ARH(1) equation (2) (see also Eq. (5) for its general definition). The real separable Hilbert space H is given by $H = L^2\left([0, h], \beta_{[0,h]}, \lambda + \delta_{(h)}\right)$, where $\beta_{[0,h]}$ is the Borel σ-algebra generated by the subintervals in $[0, h]$, λ is the Lebesgue measure, and $\delta_{(h)}(s) = \delta(s - h)$ is the Dirac measure at point h. The associated norm

$$\|f\|_{H=L^2([0,h],\beta_{[0,h]},\lambda+\delta_{(h)})} = \sqrt{\int_0^h f^2(t)dt + f^2(h)}$$

establishes the equivalent classes of functions given by the relationship $f \sim_{\lambda+\delta_{(h)}} g$ if and only if $\left(\lambda + \delta_{(h)}\right)(\{t : f(t) \neq g(t)\}) = 0$, with

$$\left(\lambda + \delta_{(h)}\right)(\{t : f(t) \neq g(t)\}) = 0 \Leftrightarrow \lambda(\{t : f(t) \neq g(t)\}) = 0 \quad \text{and} \quad f(h) = g(h), \tag{4}$$

where, as before, $\delta_{(h)}$ is the Dirac measure. We will prove, in Lemma 1, that $X = (X_n,\ n \in \mathbb{Z})$, constructed in (1) from the O.U. process, satisfying Eqs. (2)–(3), is the unique stationary solution to Eq. (2), in the space $H = L^2\left([0, h], \beta_{[0,h]}, \lambda + \delta_{(h)}\right)$, admitting a MAH($\infty$) representation. Similarly, in Lemma 4, we will prove that $X = (X_n,\ n \in \mathbb{Z})$, constructed in (1) from the O.U. process, satisfying Eqs. (2)–(3), is the unique stationary solution to Eq. (2), admitting a MAB(∞) representation, in the space $B = \mathcal{C}([0, h])$, the Banach space of continuous functions, whose support is the interval $[0, h]$, with the supremum norm.

The main results of this paper provide the almost surely convergence to ρ_θ of the MLE $\rho_{\widehat{\theta}}$ of ρ_θ, in the norm of $\mathcal{L}(H)$, the space of bounded linear operators in the Hilbert space H (respectively, in the norm of $\mathcal{L}(B)$, the space of bounded linear operators in the Banach space B). The convergence in probability of the associated plug-in ARH(1) and ARB(1) predictors, i.e., the convergence in probability of $\rho_{\widehat{\theta}}(X_{n-1})$ to $\rho_\theta(X_{n-1})$ in H and B, respectively, is proved as well.

The outline of this paper is as follows. In Section 2, the main results of this paper are obtained. Specifically, Section 2.1 provides the definition of O.U. process as an ARH(1) process. Strong consistency in $\mathcal{L}(H)$ of the estimator of the autocorrelation operator is derived in Section 2.2. Consistency in H of the associated plug-in ARH(1) predictor is then established in Section 2.3. The corresponding results in Banach spaces are given in Section 2.4. For illustration purposes, a simulation study is undertaken in Section 3. Final comments can be found in Section 4. (The basic preliminary elements applied in the proof of the main results of this paper and the proof of Lemma 1 can be found in the supplementary material, see Appendix A.)

2. Prediction of O.U. process in Hilbert and Banach spaces

In this section, we consider H to be a real separable Hilbert space. Recall that a zero-mean ARH(1) process $X = (X_n,\ n \in \mathbb{Z})$, on the basic probability space $(\Omega, \mathcal{A}, P)$, satisfies (see Bosq, 2000)

$$X_n = \rho(X_{n-1}) + \varepsilon_n, \quad n \in \mathbb{Z}, \tag{5}$$

where ρ denotes the autocorrelation operator of process X. Here, $\varepsilon = (\varepsilon_n,\ n \in \mathbb{Z})$ is assumed to be a strong-white noise, i.e., ε is a Hilbert-valued zero-mean stationary process, with independent and identically distributed components in time, and with $\sigma^2 = E\|\varepsilon_n\|_H^2 < \infty$, for all $n \in \mathbb{Z}$.