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On the multivariate skew-normal-Cauchy distribution

F. Kahrari [a,*], M. Rezaei [a], F. Yousefzadeh [a], R.B. Arellano-Valle [b]

[a] *Department of Statistics, School of Mathematical Sciences and Statistics, University of Birjand, Birjand, Iran*

[b] *Departamento of Estadística, Pontificia Universidad Católica de Chile, Santiago, Chile*

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ABSTRACT

We study some of the main probabilistic properties of the so called multivariate skew-normal-Cauchy distribution. Simple expressions to compute the entries of the expected Fisher information matrix of this multivariate distribution are proposed.

1. Introduction

The multivariate skew-normal (SN) distribution is introduced by Azzalini and Dalla Valle (1996) in terms of its probabilistic density function (pdf) as

$$2\phi_p(\mathbf{x};\, \mathbf{\Sigma})\Phi(\boldsymbol{\lambda}^\top \mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^p \tag{1}$$

where $\mathbf{\Sigma} > \mathbf{0}$ is a correlation matrix, $\boldsymbol{\lambda} \in \mathbb{R}^p$ is a skewness parameter vector and $\phi_p(\mathbf{x};\, \mathbf{\Sigma})$ is the pdf of the p-variate normal distribution with zero mean vector and correlation matrix $\mathbf{\Sigma}$, denoted by $N_p(\mathbf{0},\, \mathbf{\Sigma})$, and $\Phi(x)$ is the cumulative distribution function (cdf) of the standard univariate normal distribution. For further properties and applications of this distribution we refer the reader to Azzalini and Capitanio (1999), Genton et al. (2001), Loperfido (2001) and Azzalini and Capitanio (2003), among others. From these works, various extensions have been developed; see e.g. Branco and Dey (2001), Genton and Loperfido (2005), Arellano-valle and Genton (2005), and Arellano-Valle and Azzalini (2006). Survey paper by Azzalini (2005) and the books edited by Genton (2004) and Azzalini and Capitanio (2014) give a critical overview of research in this area. Genton and Loperfido (2005) introduced the generalized skew-normal (GSN) distribution whose pdf is as follows:

$$f_{\mathbf{X}}(\mathbf{x}) = 2\phi_p(\mathbf{x};\, \boldsymbol{\mu},\, \mathbf{\Sigma})\Pi(\mathbf{x} - \boldsymbol{\mu}), \quad \mathbf{x} \in \mathbb{R}^p \tag{2}$$

* Corresponding author.
E-mail address: fkahrari@birjand.ac.ir (F. Kahrari).

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where the skewing function Π satisfies $0 \leq \Pi(\mathbf{x}) \leq 1$ and $\Pi(-\mathbf{x}) = 1 - \Pi(\mathbf{x})$, $\forall \mathbf{x} \in \mathbb{R}^p$. Clearly, for $\Pi(\mathbf{x}) = 1/2$, $\mathbf{X}$ has a $N_p(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ distribution. For the special case by taking $\Pi(\mathbf{x}) = G(\boldsymbol{\lambda}^\top \mathbf{x})$, with G being an absolutely continuous cdf such as normal, Laplace, logistic or uniform cdf's, Huang et al. (2013) derived an explicit forms for the moment generating function (mgf) of these mentioned multivariate distributions. However, perhaps due to the complexity of the mgf of the multivariate skew-normal-Cauchy (SNC) distribution, they did not address this distribution, in which G is the cdf of the univariate Cauchy distribution denoted here by $\Psi(x) = \frac{1}{2} + \frac{1}{\pi}\arctan(x)$, $x \in \mathbb{R}$. To get around this difficulty, we extend a convenient and useful hierarchical representation of the univariate SNC distribution, which was derived by Arrue et al. (2011), to the multivariate case. This representation facilitates, for example, the computation of moments, the mgf and the multivariate skewness and kurtosis coefficients. The main objective of this work is to show that the multivariate SNC distribution preserves a number of convenient formal properties of the multivariate SN one. An important feature of the multivariate SNC distribution is that it possess wider kurtosis range than the multivariate SN distribution. Some works on the computation of the expected Fisher information matrix (FIM) of a multivariate family of skewed distributions have been developed by Arellano-Valle and Azzalini (2008), Arellano-Valle (2010), Ley and Paindaveine (2010a,b) and Hallin and Ley (2012). It would appear that no more work has been done on finding expressions for the entries of the FIM of other specific members of the multivariate GSN distributions, as far as we know. In this work, we compute the FIM of the multivariate SNC distribution.

The outline of this paper is as follows. In Section 2, we define the multivariate SNC distribution, and then we represent it as shape mixture of the multivariate SN distribution. A stochastic representation of the multivariate SNC distribution is also derived in Section 2. In Section 3, the first four moments and some summary indexes of skewness and kurtosis of a random vector with the multivariate SNC distribution are obtained. In Section 4, we study the distribution of linear transformations and quadratic forms of a random vector with multivariate SNC distribution. In Section 5, we study the marginal and conditional distributions obtained from a partition of a SNC random vector. In Section 6, convenient expressions for the entries the expected FIM of the multivariate SNC distribution are obtained, and we show that, in the vicinity of symmetry, this matrix is singular.

2. Multivariate skew-normal-Cauchy distribution

In this section, we define the multivariate SNC distribution in terms of its pdf and represent it as shape mixture of the multivariate SN distribution. The idea of shape mixture of a skew-symmetric distribution is introduced by Arellano-Valle et al. (2008), (see also Arellano-Valle et al., 2009) and is applied by Arrue et al. (2011) to the univariate SNC distribution. Following Genton and Loperfido (2005), the multivariate SNC distribution is defined as follows.

Definition 2.1. We say that a random vector $\mathbf{X} \in \mathbb{R}^p$ has a multivariate SNC distribution, with location parameter $\boldsymbol{\mu} \in \mathbb{R}^p$, dispersion matrix $\boldsymbol{\Sigma} > \mathbf{0}$, a positive definite matrix of dimension $p \times p$,with $\boldsymbol{\sigma} = \mathrm{diag}(\boldsymbol{\Sigma})^{1/2}$, and skewness vector $\boldsymbol{\lambda} \in \mathbb{R}^p$, denoted by $\mathbf{X} \sim SNC_p\,(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \boldsymbol{\lambda})$, if its pdf is given by

$$f_{\mathbf{X}}(\mathbf{x}) = 2\phi_p(\mathbf{x}; \boldsymbol{\mu}, \boldsymbol{\Sigma})\left(\frac{1}{2} + \frac{1}{\pi}\arctan\left(\boldsymbol{\lambda}^\top \boldsymbol{\sigma}^{-1}(\mathbf{x} - \boldsymbol{\mu})\right)\right), \quad \mathbf{x} \in \mathbb{R}^p. \tag{3}$$

In what follows, we assume by simplicity that $\boldsymbol{\mu} = \mathbf{0}$ and $\boldsymbol{\Sigma}$ is a correlation matrix, so that $\boldsymbol{\sigma} = \mathbf{I}_p$, the identity matrix of dimension $p \times p$.

Proposition 2.1. *If* $\mathbf{X}|Y = y \sim SN_p(\mathbf{0}, \boldsymbol{\Sigma}, \boldsymbol{\lambda}y)$ *and* $Y \sim HN(0, 1)$, *then* $\mathbf{X} \sim SNC_p(\mathbf{0}, \boldsymbol{\Sigma}, \boldsymbol{\lambda})$.

Proof. Since $f_{\mathbf{X}|Y=y}(\mathbf{x}|y) = 2\phi_p(\mathbf{x}; \mathbf{0}, \boldsymbol{\Sigma})\Phi(\boldsymbol{\lambda}^\top \mathbf{x}y)$, $\mathbf{x} \in \mathbb{R}^p$, and $f_Y(y) = 2\phi(y)$, $y > 0$, by relation 6 in Gómez et al. (2007) we have

$$\begin{aligned} f_{\mathbf{X}}(\mathbf{x}) &= 2\phi_p(\mathbf{x}; \mathbf{0}, \boldsymbol{\Sigma})\int_0^\infty 2\phi(y)\Phi(\boldsymbol{\lambda}^\top \mathbf{x}y)dy \\ &= 2\phi_p(\mathbf{x}; \mathbf{0}, \boldsymbol{\Sigma})\left(\frac{1}{2} + \frac{1}{\pi}\arctan(\boldsymbol{\lambda}^\top \mathbf{x})\right). \quad \square \end{aligned}$$

Remark 2.1. Proposition 2.1 represents the multivariate SNC distribution as a shape mixture of a multivariate SN distribution with a univariate half-normal mixing distribution.

Corollary 2.1. *If* $\mathbf{X} \sim SNC_p(\mathbf{0}, \boldsymbol{\Sigma}, \boldsymbol{\lambda})$, *then*

i. *For every even function* g, $g(\mathbf{X}) \stackrel{d}{=} g(\mathbf{X}_0)$, *where* $\mathbf{X}_0 \sim N_p(\mathbf{0}, \boldsymbol{\Sigma})$