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Expected distances and goodness-of-fit for the asymmetric Laplace distribution

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ABSTRACT

New results on expected distances for the asymmetric Laplace distribution are derived and applied to develop a new goodness-of-fit test for the asymmetric Laplace distribution based on the energy distance.

1. Introduction

The classical Laplace distribution with location parameter $\theta \in \mathbb{R}$ and scale parameter $s > 0$ is the distribution with probability density function

$$f(x; \theta, s) = \frac{1}{2s} e^{-|x-\theta|/s}, \quad -\infty < x < \infty. \tag{1.1}$$

The standard Laplace distribution is the special case $\theta = 0$ and $s = 1$, which has variance 2. For an interesting historical background on the development of the classical Laplace distribution, see Kotz et al. (2001, Ch. 1), who also collect and present applications aimed to illustrate that the Laplace distribution and its generalizations in some cases can be a useful alternative to the normal law.

A generalization of the Laplace family of distributions is the asymmetric Laplace, a skewed family of distributions that contains the Laplace distributions as a special case. We follow Definition 3.1.1 of Kotz et al. (2001).

Definition 1. A random variable Y has an asymmetric Laplace distribution $\mathcal{AL}(\theta, \mu, \sigma)$ if there exist parameters $\theta \in \mathbb{R}$, $\mu \in \mathbb{R}$, and $\sigma \geq 0$ such that the characteristic function (cf.) of Y has the form

$$\psi(t) = \frac{e^{i\theta t}}{1 + \frac{1}{2}\sigma^2 t^2 - i\mu t}, \quad -\infty < t < \infty. \tag{1.2}$$

When $\mu = 0$ (1.2) is the c.f. of a symmetric Laplace distribution.

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In Section 2 we present some new results on expected distances. These results are applied to developing the energy goodness-of-fit test for asymmetric Laplace distribution in Section 3.

Laplace and asymmetric Laplace distributions have been applied to model some types of heavy tailed data; for example, stock market returns, currency exchange rates, or non-Gaussian noise in signal analysis (Kotz et al., 2001, Ch. 7). The class of energy tests (Székely and Rizzo, 2013) is attractive for this type of goodness-of-fit problem because energy tests typically perform quite well when the null or alternative distribution is heavy tailed. Another advantage of the energy test is that it has a natural extension to testing for bivariate or multivariate Laplace distributions.

2. Expected distances for asymmetric Laplace distributions

In this section, the expected distance $E|y - Y|$ of an asymmetric Laplace random variable Y to a point y, and expected distance $E|Y - Y'|$ between two independent and identically distributed (i.i.d.) asymmetric Laplace variables Y and Y' are derived. Throughout this paper $|\cdot|$ denotes the Euclidean norm or absolute difference.

For deriving our results, we work with the following alternate parameterization. We write $Y \sim \mathcal{AL}^*(\theta, \kappa, \sigma)$ (see Kotz et al., 2001, Sec. 3.1) if Y has the density function

$$f_{\theta,\kappa,\sigma}(y) = \frac{\sqrt{2}}{\sigma}\frac{\kappa}{1+\kappa^2}\begin{cases}\exp\left(-\frac{\sqrt{2}\kappa}{\sigma}|y-\theta|\right), & y \geq \theta;\\ \exp\left(-\frac{\sqrt{2}}{\sigma\kappa}|y-\theta|\right), & y < \theta.\end{cases} \tag{2.1}$$

The skewness parameters μ and κ are related by

$$\kappa = \frac{\sqrt{2}\,\sigma}{\mu + \sqrt{2\sigma^2+\mu^2}} = \frac{\sqrt{2\sigma^2+\mu^2}-\mu}{\sqrt{2}\,\sigma}, \qquad \mu = \frac{\sigma}{\sqrt{2}}\left(\frac{1}{\kappa} - \kappa\right). \tag{2.2}$$

If $\mu = 0\,(\kappa = 1)$ the density (2.1) is a symmetric Laplace density. The standard Laplace case is $\mathcal{AL}^*(\theta = 0, \kappa = 1, \sigma = \sqrt{2})$.

The cumulative distribution function (c.d.f.) of $Y \sim \mathcal{AL}^*(\theta, \kappa, \sigma)$ is

$$F_{\theta,\kappa,\sigma}(y) = \begin{cases}1 - \frac{1}{1+\kappa^2}\exp\left(-\frac{\sqrt{2}\kappa}{\sigma}|y-\theta|\right), & y \geq \theta;\\ \frac{\kappa^2}{1+\kappa^2}\exp\left(-\frac{\sqrt{2}}{\sigma\kappa}|y-\theta|\right), & y < \theta.\end{cases} \tag{2.3}$$

Introduce

$$p_\kappa := Pr(Y > \theta) = \frac{1}{1+\kappa^2}, \qquad q_\kappa := Pr(Y \leq \theta) = \frac{\kappa^2}{1+\kappa^2},$$

and

$$\lambda := \frac{\sqrt{2}\kappa}{\sigma}, \qquad \beta := \frac{\sqrt{2}}{\kappa\sigma}.$$

In this notation the density is

$$f_{\theta,\kappa,\sigma}(y) = \begin{cases}\lambda p_\kappa \exp(-\lambda|y-\theta|), & y \geq \theta;\\ \lambda p_\kappa \exp(-\beta|y-\theta|), & y < \theta,\end{cases} \tag{2.4}$$

and the c.d.f. is

$$F_{\theta,\kappa,\sigma}(y) = \begin{cases}1- \quad p_\kappa \exp(-\lambda|y-\theta|), & y \geq \theta;\\ \qquad q_\kappa \exp(-\beta|y-\theta|), & y < \theta.\end{cases} \tag{2.5}$$

The mean and variance of $Y \sim \mathcal{AL}^*(\theta, \kappa, \sigma)$ are

$$E[Y] = \theta + \frac{\sigma}{\sqrt{2}}\left(\frac{1}{\kappa} - \kappa\right) = \theta + \mu, \qquad Var[Y] = \frac{\sigma^2}{2}\left(\frac{1}{\kappa^2} + \kappa^2\right) = \mu^2 + \sigma^2.$$

2.1. Expected distance $E|y - Y|$ for an arbitrary point y

Kotz et al. (2001, (3.1.26)) derived the expected distance of $Y \sim \mathcal{AL}^*(\theta, \kappa, \sigma)$ to the parameter θ as

$$E|Y - \theta| = \frac{\sigma}{\sqrt{2}\kappa}\frac{1+\kappa^4}{1+\kappa^2}.$$