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# Ruin probabilities under Sarmanov dependence structure

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ABSTRACT

Our work aims to study the tail behaviour of weighted sums of the form $\sum_{i=1}^{\infty} X_i \prod_{j=1}^{i} Y_j$, where $(X_i, Y_i)$ are independent and identically distributed, with common joint distribution bivariate Sarmanov. Such quantities naturally arise in financial risk models. Each $X_i$ has a regularly varying tail. With sufficient conditions similar to those used by Denisov and Zwart (2007) imposed on these two sequences, and with certain suitably summable bounds similar to those proposed by Hazra and Maulik (2012), we explore the tail distribution of the random variable $\sup_{n\geq 1} \sum_{i=1}^{n} X_i \prod_{j=1}^{i} Y_j$. The sufficient conditions used will relax the moment conditions on the $\{Y_i\}$ sequence.



## 1. Introduction

Regularly varying distributions find several applications in areas of actuarial and financial mathematics, in the analysis of random coefficient linear processes such as ARMA and FARIMA, and in stochastic difference equations. We refer to Nyrhinen (2012) for the study of the insurance ruin problem. The development of the capital is described as the solution to a stochastic difference equation. The net losses over the years are independent and identically distributed with regularly varying tail. Yang and Wang (2013) consider a discrete-time risk model with dependent insurance and financial risks. If $X_n$ denotes the insurance risk and $Y_n$ the financial risk or the stochastic discount factor in time $n$, then

$$S_n = \sum_{i=1}^{n} X_i \prod_{j=1}^{i} Y_j \tag{1}$$

represents the stochastic discount value of aggregate net losses up to time $n$. In Yang and Wang (2013), the finite and infinite time ruin probabilities are analysed.

A random variable $X$ with tail distribution $\overline{F}$ is said to be regularly varying with index $-\alpha$, with $\alpha > 0$, if $\overline{F}(xy) \sim y^{-\alpha}\overline{F}(x)$ as $x \to \infty$, for all $y > 0$. This is denoted by $X \in RV_{-\alpha}$. Let $\{X_n, n \geq 1\}$ be a sequence of independent and identically distributed random variables with regularly varying tails, and $\{\Theta_n, n \geq 1\}$ be another sequence of random variables, not necessarily independent of $\{X_n\}$. The almost sure convergence and tail behaviour of $\sup_{n\geq 1} \sum_{i=1}^{n} \Theta_i X_i$ has been studied in the literature. Here and later, for two positive functions $a(x)$ and $b(x)$, we write $a(x) \sim b(x)$ as $x \to \infty$ if $\lim_{x\to\infty} a(x)/b(x) = 1$.

The study of the almost sure finiteness of the infinite sum $S_\infty = \sum_{i=1}^{\infty} X_i \prod_{j=1}^{i} Y_j$ has been a topic of sustained interest in the literature. The general problem has been addressed in Hult and Samorodnitsky (2008) for the case when the sequences $\{X_i\}$ and $\{Y_i\}$ are independent and $\{X_i\}$ an i.i.d. regularly varying sequence. See also Fougeres and Mercadier (2012) and Yang and Hashorva (2013).

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We address our problem in two parts: first we analyse the behaviour of the product, and then the sum. The main result in this direction is given in Breiman (1965), which proves that if $X \in RV_{-\alpha}$ and $\Theta$ independent of $X$ satisfies $E[\Theta^{\alpha+\varepsilon}] < \infty$ for some $\varepsilon > 0$, then $\Theta X \in RV_{-\alpha}$ with $P[\Theta X > x] \sim E[\Theta^{\alpha}]P[X > x]$ as $x \to \infty$. This result was extended to finite and infinite sums in Resnick and Willekens (1991). They showed that if $\{X_i\}$ and $\{\Theta_i\}$ are independent of each other, the $X_i$s are i.i.d $RV_{-\alpha}$, and the $\Theta_i$s satisfy some extra moment assumptions, then $P\left[\sum_{i=1}^{\infty} \Theta_i X_i > x\right] \sim P[X_1 > x] \sum_{i=1}^{\infty} E[\Theta_i^{\alpha}]$ as $x \to \infty$.

Denisov and Zwart (2007) replaced the extra moment assumptions with other sufficient conditions so that $P[\Theta X > x] \sim E[\Theta^{\alpha}]P[X > x]$ as $x \to \infty$. This was again extended to the finite and infinite sum case by Hazra and Maulik (2012). Motivated by the ruin model of Nyrhinen (2012), we restrict ourselves to the setup where $\Theta_i = \prod_{j=1}^{i} Y_j$, for i.i.d. $Y_j$.

We consider the finite time ruin probability by time $n$, given by

$$\Psi(x, n) = P\left[\max_{1\le k\le n} S_k > x\right], \tag{2}$$

and the infinite time ruin probability by

$$\Psi(x) = P\left[\sup_{n\ge 1} S_n > x\right]. \tag{3}$$

### 1.1. Some useful classes of distributions

While classically, the insurance risk $\{X_n\}$ and discount factor $\{Y_n\}$ are assumed to be independent, Yang and Wang (2013) assumed that each $(X_i, Y_i)$ follows a bivariate Sarmanov distribution, which is defined as follows.

**Definition 1.1.** The pair of random variables $(X, Y)$ is said to follow a bivariate Sarmanov distribution, if

$$P(X \in dx, Y \in dy) = (1 + \theta\phi_1(x)\phi_2(y))F(dx)G(dy), \quad x \in \mathbb{R}, y \ge 0,$$

where the kernels $\phi_1$ and $\phi_2$ are two real valued functions and the parameter $\theta$ is a real constant satisfying

$$E\{\phi_1(X)\} = E\{\phi_2(Y)\} = 0 \quad \text{and} \quad 1 + \theta\phi_1(x)\phi_2(y) \ge 0, \quad x \in D_X, y \in D_Y,$$

where $D_X \subset \mathbb{R}$ and $D_Y \subset \mathbb{R}^+$ are the supports of $X$ and $Y$, with marginals $F$ and $G$ respectively.

This class of bivariate distributions is quite wide, covering a large number of well-known copulas such as the Farlie–Gumbel–Morgenstern (FGM) copula, which is recovered by taking $\phi_1(x) = 1 - 2F(x)$ and $\phi_2(y) = 1 - 2G(y)$. We refer the reader to Ting Lee (1996) for further discussion. A bivariate Sarmanov distribution is called proper if $\theta \neq 0$ and none of $\phi_1$ and $\phi_2$ vanishes identically. To study the dependence structure of Sarmanov distribution, we need to define the class of dominatedly tail varying distributions.

**Definition 1.2.** A random variable $X$ with distribution function $F$ is called dominatedly-tail-varying, denoted by $X \in \mathcal{D}$, if for all $0 < y < 1$, $\limsup_{x\to\infty} \overline{F}(xy)/\overline{F}(x) < \infty$.

It is traditional to study the tail of the product of random variables in terms of the Breiman's condition, which we strive to weaken. For that we need to state definitions of certain useful classes of distributions.

**Definition 1.3.** A random variable $X$ is said to be long tailed and denoted by $X \in \mathcal{L}$ if $P[X > x] \sim P[X > x + y]$ as $x \to \infty$, for any $y$.

**Definition 1.4.** A non-negative function $f$ is in the class $\mathcal{S}_d$ and called a subexponential density if

$$\lim_{x\to\infty} \int_0^x \frac{f(x-y)}{f(x)} f(y)dy = 2\int_0^{\infty} f(u)du < \infty.$$

If $f \in \mathcal{S}_d$ is such that $f(x) = P[U > x]$ for some random variable $U$, we say that $U \in \mathcal{S}^*$.

**Definition 1.5.** A non-negative random variable $T$ is in class $\mathcal{S}(\gamma)$, $\gamma \ge 0$, if as $x \to \infty$, we have

$$\frac{P[T > x + y]}{P[T > x]} \to e^{-\gamma y} \quad \text{and} \quad \frac{P[T + T' > x]}{P[T > x]} \to 2E[e^{\gamma T}] < \infty,$$

where $T'$ is an i.i.d. copy of $T$. For $\gamma = 0$, we get the class $\mathcal{S}$ of subexponential distributions.