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Small deviation probabilities for weighted sum of independent random variables with a common distribution that can decrease at zero fast enough

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ABSTRACT

We examine small deviation probabilities of weighted sum of independent random variables with a common distribution that can decrease at zero fast enough and satisfies mild moment assumptions.

1. Introduction and results

Let $S = \sum_{j\geq 1} \lambda(j) X_j$, where $\{X_i\}$ are independent copies of a positive random variable X with a distribution function $V(x)$, and let $\lambda(\cdot)$ be a bounded positive *non-increasing* function defined on the interval $[1, \infty]$. We assume that the series S converges almost surely, or

$$\sum_{j\geq 1} m(\lambda(j)) < \infty \quad \big(m(s) = \mathbf{E}\min(1, sX)\big). \tag{1.1}$$

Asymptotic behavior of probability $\mathbf{P}(S < r)$ as $r \searrow 0$ is essentially defined by the behavior of the distribution V at zero and at infinity. Remind that the mildest moment assumptions are given by condition (1.1).

If, say, $\log(1/\lambda(j)) \asymp j^c$ for some positive c then (1.1) is equivalent to condition $\mathbf{E}\log^{1/c}(1+X) < \infty$.

Here and in what follows

$$u(y) \asymp v(y) \iff \ln\{u(y)/v(y)\} = O(1), \quad y \to \infty \text{ or } 0.$$

If we do not restrict behavior V at zero too much, it is possible to obtain an optimal *logarithmic* asymptotics for $\mathbf{P}(S < r)$ (for instance, Aurzada, 2007, Aurzada, 2008, Borovkov and Ruzankin, 2008a, Rozovsky, 2009, Rozovsky, 2011, Rozovsky, 2014b).

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In order to have more exact bounds, it is usually assumed that V decreases at zero as a power (is regularly varying, say) as in Borovkov and Ruzankin (2008b), Davis and Resnick (1991), Rozovsky (2012) and Rozovsky (2015b), or satisfies more mild restrictions as in Dunker et al. (1998), Lifshits (1997), Rozovsky (2007), Rozovsky (2014a), Rozovsky (2014c) or Rozovsky (2015a):

L. *There exist constants $b \in (0,1)$, $c_1, c_2 > 1$ and $\varepsilon > 0$ such that for each $r \le \varepsilon$*

$$c_1\,V(b\,r) \le V(r) \le c_2\,V(b\,r) \tag{1.2}$$

or

R. *There exist constants $b \in (0,1)$, $c_1 > b$, $c_2 > 1$ and $\varepsilon > 0$ such that for each $r \le \varepsilon$*

$$c_1\,\nu(b\,r) \le \nu(r) \le c_2\,\nu(b\,r), \qquad \nu(y) = \frac{1}{y}\int_0^y u\,dV(u). \tag{1.3}$$

Condition **R** is weaker than **L** (it is shown in Rozovsky (2015a) that $\mathbf{L} \Longleftrightarrow \mathbf{R}\Big|_{c_1>1}$) and, in particular, allows $V(\cdot)$ to be a slowly varying function at zero.

Let us formulate the results from Rozovsky (2014a) and Rozovsky (2014c) which are essentially generalized in the present note.

Set for $u \ge 0$

$$v(u) = \mathbf{E}\,e^{-uX}, \qquad f(u) = \log v(u), \qquad L(u) = \sum_{n\ge 1} f(u\lambda(n)), \qquad \tau^2(u) = u^2\,L''(u.) \tag{1.4}$$

Theorem 1 (*Rozovsky, 2014c*)**.** *Let conditions* **R**, *(1.1) and*

$$\limsup_{s\to+0} s\,|v'''(s)|/v''(s) < \infty \tag{1.5}$$

be satisfied.

Then

$$\mathbf{P}(S < r) \asymp \frac{\exp(L(u)+ur)}{1+\tau(u)}, \quad r \to 0, \tag{1.6}$$

where function $u = u(r)$ is a unique solution of the equation

$$L'(u) + r = 0. \tag{1.7}$$

Observe that (1.5) is equivalent to the assumption that for some k the function $s^k\,v''(s)$ grows in a positive vicinity of zero, and does not restrict too much a rate of decreasing of $1 - V(\cdot)$ at infinity. It is also important to mention that asymptotics (1.6) hold without any *additional* restrictions on the behavior of the weights $\lambda(\cdot)$.

Theorem 2 (*Rozovsky, 2014a*)**.** *Let conditions* **L** *and (see (1.1))*

G. $\limsup_{n\to\infty} \sum_{l\ge 1} m(\lambda(l\,n)/\lambda(n)) < \infty$

hold.

Then

$$\mathbf{P}(S < r) \asymp \frac{\exp(L(u)+ur)}{\sqrt{\lambda^{-1}(1/u)}}, \quad r \to 0, \tag{1.8}$$

where $\lambda^{-1}(x) = \sup\{y : \lambda(y) \ge x\}$ denotes the inverse function of λ and u satisfies (1.7).

Recall that the assumption **G** often coincides with the *necessary* condition (1.1) (for example, if $\lambda(n) \asymp e^{-g(\log n)}$, where function $g(y)/y$ does not decrease at infinity). Besides that (see Rozovsky, 2014a for the details), under conditions of Theorem 2

$$\tau^2(u) \asymp \lambda^{-1}(1/u), \quad u \to \infty, \tag{1.9}$$

whence it follows, in particular, that if **L** holds then relations (1.6) and (1.8) are identical and take place both under condition (1.5) (and (1.1)), and under **G**.

Assumptions of Theorems 1 and 2 imply that rather exact asymptotics (1.6) (or (1.8)) hold if the rate of decreasing of $V(r)$ at zero does not exceed a positive degree of r. Otherwise, as it was already mentioned above, we can apply logarithmic asymptotics only.

Let us show now that (1.6) can still be true if conditions **L** or **R** are not valid.