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Deconvolution model with fractional Gaussian noise: A minimax study

Rida Benhaddou *

Department of Mathematics, Ohio University, Athens, OH 45701, United States

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ABSTRACT

We consider the problem of estimating a function in a deconvolution model with fractional Gaussian noise. We derive minimax lower and upper bounds to show that our estimator attains optimal or near optimal rates. Such rates are affected by LRD.

1. Introduction

Consider the problem of estimating a periodic function $f(t)$ with $t \in [0, 1]$, based on observations from the noisy convolution

$$Y(t) = \int_0^1 f(s)g(t-s)ds + \varepsilon^\alpha Z^H(t), \quad t \in [0, 1]. \tag{1.1}$$

Here, the function $g(\cdot)$ is known and $Z^H(t)$ is a fractional Gaussian noise, i.e., a Gaussian process with covariance function

$$\mathbb{E}\left[Z^H(t_1)Z^H(t_2)\right] = 1/2\left(|t_1|^{2H} + |t_2|^{2H} - |t_1 - t_2|^{2H}\right)$$

where $\alpha = 2 - 2H \in (0, 1]$ is the level of long-range dependence and H is Hurst parameter.

Deconvolution model has been the subject of a great deal of papers since late 1980s, but the most significant contribution was that of Donoho (1995) who was the first to devise a wavelet solution to the problem. In these attempts, it is assumed that errors are i.i.d. Gaussian random variables. However, empirical evidence has shown that even at large lags, the correlation structure in the errors can decay at a hyperbolic rate. To account for this, quite a few papers on long-range dependence (LRD) in general have been developed.

In the deconvolution model, Wishart (2013) extends the idea of Kulik and Raimondo (2009) to establish equivalence between a deconvolution model with fractional Brownian motion (fBm) and another deconvolution model with white noise. Wang (1997) studies similar problem but for a more general noisy linear transformation when the noise is fBm. In a multichannel functional deconvolution model, Benhaddou et al. (2014) consider the model under LRD errors setting without specifying the form of LRD but under the assumption that error structure satisfies certain conditions. Kulik et al. (2015) investigate a multichannel deconvolution model with LRD errors in the L^p-risk.

* Fax: +1 740 593 9805.
E-mail address: benhaddo@ohio.edu.

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The objective of this paper is to investigate the deconvolution model from the minimax point of view under the fractional Gaussian noise (fGn) error structure. The motives behind working the problem with fGn error structure stem from the fact that under such setting, the minimax results are sharper than under other forms of LRD. Following Wishart approach, Wavelet-Vaguelette-Decomposition (WVD) via bounded bandwidth wavelets is performed to de-correlate fGn. The lower bounds are to be constructed in the L^2-risk for both regular-smooth and super-smooth convolutions to be complemented by the upper bounds. In addition, we propose an adaptive estimator and show that it is asymptotically optimal or near-optimal, in a wide range of Besov balls. It is also shown that, under the regular-smooth convolution, the convergence rates are affected by long memory. In particular, the rates deteriorate as the dependence gets more severe (the Hurst parameter increases).

The rest of the paper is organized as follows. In Section 2, we describe the construction of the wavelet estimator for $f(t)$. In Section 3, we derive minimax lower bounds for the L^2-risk when $f(t)$ is assumed to belong to a Besov ball and $g(t)$ possesses some smoothness properties. In Section 4, we demonstrate that our estimator is asymptotically optimal or near-optimal within a logarithmic factor. Finally, Section 5 contains the proofs of the theoretical results.

2. Estimation algorithm

In what follows, $\langle \cdot, \cdot \rangle$ denotes the inner product in the Hilbert space $L^2([0, 1])$ (the space of squared-integrable functions defined on the unit interval $[0, 1]$), i.e., $\langle f, g\rangle = \int_0^1 f(t)\overline{g(t)}dt$ for $f, g \in L^2([0, 1])$. We also denote the complex conjugate of a by $\bar{a}$. Let $e_m(t) = e^{i2\pi mt}$ be a Fourier basis on the interval $[0, 1]$. Let $Y_m = \langle e_m, Y\rangle$, $Z_m^H = \langle e_m, Z^H\rangle$, $g_m = \langle e_m, g\rangle$ and $f_m = \langle e_m, f\rangle$ be Fourier coefficients of the functions Y, Z_H, g and f respectively. Then, applying the Fourier transform to Eq. (1.1), one obtains

$$Y_m = g_m f_m + \varepsilon^{\alpha} Z_m^H. \tag{2.2}$$

Consider a bounded bandwidth periodized wavelet basis (e.g., Meyer-type) $\psi_{j,k}(t)$. Let m_0 be the lowest resolution level for the wavelet basis and denote the scaling function for the bounded bandwidth wavelet by $\psi_{m_0-1,k}(t)$. Then, using the periodized Meyer wavelet basis described above, any periodic function $f(t) \in L^2([0, 1])$ can be expanded into a wavelet series as

$$f(t) = \sum_{j=m_0-1}^{\infty} \sum_{k=0}^{2^j-1} \beta_{j,k}\psi_{j,k}(t) \tag{2.3}$$

where $\beta_{j,k} = \langle f, \psi_{j,k}(t)\rangle$. Notice that expansion (2.3) is valid under condition (3.12) in that it guarantees the existence of an isometry between the space of squared-integrable functions and the space of summable sequences. If $\psi_{j,k,m} = \langle e_m, \psi_{j,k}\rangle$ are Fourier coefficients of $\psi_{j,k}(t)$, then Plancherel's formula and (2.2) yield the unbiased estimator

$$\tilde{\beta}_{j,k} = \sum_{m\in W_j} \frac{Y_m}{g_m}\overline{\psi}_{j,k,m} \tag{2.4}$$

where, for any $j \ge m_0$,

$$W_j = \{m : \psi_{j,k,m} \neq 0\} \subseteq 2\pi/3\left[-2^{j+2}, -2^j\right] \cup \left[2^j, 2^{j+2}\right], \tag{2.5}$$

due to the fact that Meyer wavelets are bandlimited (see, e.g., Johnstone et al., 2004, Section 3.1). We now construct a hard thresholding estimator of $f(t)$ as

$$\widehat{f}_{\varepsilon}(t) = \sum_{j=m_0-1}^{J-1} \sum_{k=0}^{2^j-1} \tilde{\beta}_{j,k}\mathbb{I}\left(|\tilde{\beta}_{j,k}| > \lambda_{j\varepsilon}^{\alpha}\right)\psi_{j,k}(t), \tag{2.6}$$

and the values of J, m_0 and $\lambda_{j\varepsilon}^{\alpha}$ will be defined later. We assume that the function $g(t)$ satisfies the following condition.

Assumption 1. Assume that the Fourier coefficients g_m of the function $g(t)$, for some positive constants ν_1, ν_2, β, C_1 and C_2, and $a_1, a_2 \ge 0$, independent of m, are such that

$$C_1|m|^{-2\nu_1}\exp\{-2a_1|m|^{\beta}\} \le |g_m|^2 \le C_2|m|^{-2\nu_2}\exp\{-2a_2|m|^{\beta}\} \tag{2.7}$$

where either $a_1a_2 > 0$ or $a_1 = a_2 = 0$ and $\nu_1 = \nu_2 = \nu > 0$. Eq. (1.1) is referred to as the regular-smooth convolution when $a_1 = a_2 = 0$, and super-smooth convolution when $a_1a_2 > 0$.

To determine the choices of J, m_0 and $\lambda_{j\varepsilon}^{\alpha}$ in (2.6), it is necessary to evaluate the variance of (2.4).