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Large deviations of mean-field stochastic differential equations with jumps

Yujie Cai [a], Jianhui Huang [b,*], Vasileios Maroulas [c,d]

[a] School of Mathematics and Statistics, Wuhan University, China
[b] Department of Applied Mathematics, The Hong Kong Polytechnic University, Hong Kong
[c] Department of Mathematics, University of Tennessee, Knoxville, TN 37996, USA
[d] Department of Mathematical Sciences, University of Bath, Claverton Down, Bath BA2 7AY, United Kingdom

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ABSTRACT

The mean-field stochastic differential equation (MFSDE) has found various applications in science and engineering. Here, we investigate a class of MFSDE with jumps, governed by a finite dimensional Brownian motion and a Poisson random measure. We study large deviation estimates of its path solution and our approach for verifying the large deviation principle is based on the weak convergence arguments.

1. Introduction

The study of mean-field stochastic differential equation (MFSDE) arises naturally from various research topics. A general form of a MFSDE can be represented as follows:

$$dX_t = b\left(t, X_t, \mathbb{E}\varphi(X_t)\right) dt + \sigma\left(t, X_t, \mathbb{E}\varphi(X_t)\right) dW_t, \tag{1.1}$$

where W_t is a multi-dimensional Brownian motion defined on a pertinent probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We remark that the coefficients of Eq. (1.1) depend on the state law of the solution through some appropriate function φ. There are several studies on the MFSDE and the reader may refer to the following partial list for example Bossy and Talay (1997), Buckdahn et al. (2009), Dawson (1983), Kotelenez (1995), Lasry and Lions (2007) and Pra and Hollander (1995). A classical example of an MFSDE is the McKean–Vlasov model from chemistry kinetics and statistical mechanics (e.g., see Chan, 1994; McKean, 1966; Sznitman, 1991; Talay and Vaillant, 2003 and the references therein). Another area for which the MFSDEs are applicable is the study of large-population particle systems which are well formulated in many different scientific fields such as economics, biology, engineering, social science and finance (e.g., Garnier et al., 2013; Lambson, 1984; Lasry and Lions, 2007).

* Corresponding author.
E-mail address: majhuang@polyu.edu.hk (J. Huang).

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To be more precise, let us consider the following large population particle systems, driven by a small noise Brownian motion,

$$dx_t^{i,n} = b\left(t, x_t^{i,n}, \frac{1}{n}\sum_{j=1}^{n}\varphi(x_t^{j,n})\right)dt + \sqrt{\varepsilon}\sigma\left(t, x_t^{i,n}, \frac{1}{n}\sum_{j=1}^{n}\varphi(x_t^{j,n})\right)dW_t^i. \tag{1.2}$$

Letting the particle number $n \to \infty$, and considering the associated mean square limit, Eq. (1.2) will converge to a small noise MFSDE similar to (1.1), where σ therein is replaced by $\sqrt{\varepsilon}\sigma$.

On the other hand, in many scenarios, the individual particles and the related whole population will demonstrate some sudden jumps which can be formulated by a Poisson random measure. Motivated by such considerations, we focus on the small parameter MFSDE driven by both a Brownian motion and a Poisson random measure (see Eq. (3.8)) which act independently. Specifically, we show a uniform large deviation principle (ULDP) which is equivalent with the uniform Laplace principle in Polish space settings (e.g., Dembo and Zeitouni, 2009; Dupuis and Ellis, 1997; Budhiraja et al., 2008). This type of uniform estimate is fruitful for the study of the exit time problem of MFSDEs, which in turn is connected to the asymptotic behavior of dynamic systems consisting of a large number of negligible "players" with competing behaviors. For instance, in the production planning where the price of product depends on the average of considerable small production firms, the individual production is modeled by some small noise SDE. If one wants to study the production's behavior (and its associated rare events) given that the production experiences drastic fluctuations as time evolves, then the verification of such large deviation estimates is required. In addition, our study of LDP with *small noise* is also different from that of the large population (e.g., Budhiraja et al., 2012) as the LDP studied there is based on the asymptotic property of players' *number*. Moreover, the studies in Feng (1994a,b), Léonard (1995) discuss an LDP of mean-field interacting particle systems with jumps only, however, their techniques are not suitable when the nonlinearity is considered in the variance coefficient like in our scenario herein.

The main step of demonstrating large deviations is a variational representation addressed in Budhiraja et al. (2011). The key components of this variational representation are the controlled Poisson random measure and the controlled Brownian motion. The controlled Brownian motion basically shifts the mean, while the controlled Poisson random measure plays the role of a thinning function. More precisely, for establishing the uniform LDP (Theorem 3.2), one may consider controlled analogues of the mean-field SDE, and establish a weak form of asymptotic continuity as it is expressed in Condition 2.1. The reader should be reminded that this weak convergence approach to large deviations was firstly used in a finite dimensional setting in Boué and Dupuis (1998). Next, a large deviation principle was established in Budhiraja et al. (2008) for the solution of a stochastic partial differential equation driven by an infinite dimensional Brownian motion. The large deviation estimates were based on a variational representation proposed in Budhiraja and Dupuis (2000). This machinery has been extensively used in other works, e.g., Duan and Millet (2009), Manna et al. (2009), Maroulas and Xiong (2013), Sritharan and Sundar (2006) and Wang and Duan (2009).

The outline of the paper is as follows. Section 2 discusses the preliminary results on large deviation principle and its equivalent Laplace principle and some notational conventions as well. The main result of the uniform large deviation estimates of the solution of a mean-field SDE is presented in Section 3, and its proof can be found in Section 4.

2. Preliminary results and notations

2.1. Notational conventions

Consider a locally compact Polish space $\mathbb{X}$, we denote by $\mathcal{M}(\mathbb{X})$ the space of all measures ν on $(\mathbb{X}, \mathcal{B}(\mathbb{X}))$, satisfying $\nu(B) < \infty$ for every compact subset B of $\mathbb{X}$. Here, $\mathcal{B}(\mathbb{X})$ is the Borel σ-field on $\mathbb{X}$. Denote $\mathbb{C}(\mathbb{X})$ as the space of continuous functions on $\mathbb{X}$. Moreover, we endow it with the weakest topology such that for every $f \in \mathbb{C}_c(\mathbb{X}) = \{f \in \mathbb{C}(\mathbb{X}) : f \text{ has compact support}\}$, the function $\nu \to \langle f, \nu\rangle = \int_{\mathbb{X}} f(u)\nu(du)$, $\nu \in \mathcal{M}(\mathbb{X})$ is continuous. Furthermore, $\mathcal{M}(\mathbb{X})$ is a Polish space under a pertinent metric, e.g. see Maroulas (2011). Moreover, the corresponding Borel σ-field on $\mathcal{M}(\mathbb{X})$ under this topology will be denoted by $\mathcal{B}(\mathcal{M}(\mathbb{X}))$. Now we present the definition of a Poisson random measure (PRM) as follows.

Definition 2.1. Let $(A, \mathcal{A}, \mu)$ be some measure space with σ-finite measure μ. The Poisson random measure with intensity measure μ is a family of random variables $\{N(B), B \in \mathcal{A}\}$ defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ such that

1. $\forall \omega \in \Omega,\ N(\cdot, \omega)$ is a measure on $(A, \mathcal{A})$.
2. $\forall B \in \mathcal{A},\ N(B)$ is a Poisson random variable with rate $\mu(B)$, i.e., $\mathbb{P}(N(B) = n) = \frac{e^{-\mu(B)}}{n!}\mu(B)^n$.
3. If $B_1, B_2, \ldots, B_n \in \mathcal{A}$ are disjoint, then $N(B_1), N(B_2), \ldots, N(B_n)$ are mutually independent.

Our problem treats large deviations for a mean-field type SDE whose solution eventually can be written as a measurable image of maps applied on a PRM and a Brownian motion. Taking into consideration the aforementioned framework, we proceed by describing the appropriate probability space as follows. Denote by $\mathbb{W} = \mathbb{C}([0, T]; \mathbb{R}^d)$ the space of all continuous functions from $[0, T]$ to $\mathbb{R}^d$ and $\mathbb{M} = \mathcal{M}(\mathbb{X}_{\mathbb{T}})$ the space of measures on $\mathbb{X}_T = [0, T] \times \mathbb{X}$ for a fixed $T > 0$. Take $\mathbb{V} = \mathbb{W} \times \mathbb{M}$ and let $\mathbb{P}^\theta$ be the unique probability measure on $(\mathbb{V}, \mathcal{B}(\mathbb{V}))$ such that (i) $N : \mathbb{V} \to \mathbb{M}$ is a PRM with intensity $\theta\nu_T$, where $\nu_T = \lambda_T \otimes \nu,\ \lambda_T$ is the Lebesgue measure on $[0, T]$ and $\nu(B) < \infty$ for all $B \in \mathcal{B}(\mathbb{X}_T)$; (ii) $W : \mathbb{V} \to \mathbb{W}$ is a $\mathbb{R}^d$-valued