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Backward doubly stochastic differential equations with stochastic Lipschitz condition

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ABSTRACT

We prove an existence and uniqueness result for backward doubly stochastic differential equations whose coefficients satisfy a stochastic Lipschitz condition. A comparison theorem for stochastic Lipschitz is also proved.

1. Introduction

Nonlinear backward stochastic differential equations (BSDEs in short) were first introduced by Pardoux and Peng (1990) in 1990 with uniform Lipschitz condition under which they proved the celebrated existence and uniqueness result. Since then, the theory of BSDEs has been intensively developed in the last years. The great interest in this theory comes from its connections with many other fields of research, such as mathematical finance (see EL Karoui et al., 1997 and its references), stochastic control, stochastic games (see Hamadène and Lepeltier, 1995a, Hamadène and Lepeltier, 1995b) and partial differential equations (PDEs) (see Pardoux and Peng, 1992 and Peng, 1991). Unfortunately, in many applications, the usual Lipschitz conditions cannot be satisfied. For example, the pricing of an european claim is equivalent to solve the linear BDSE

$$\begin{cases} -dY_t = [r(t)Y_t + \theta(t)Z_t]dt - Z(t)dW_t \\ Y_T = \xi \end{cases} \tag{1.1}$$

where ξ is the contingent claim, $r(t)$ is the interest rate and $\theta(t)$ is the risk premium vector. Both $r(t)$ and $\theta(t)$ are not bounded in general. So, it is not possible to solve Eq. (1.1) by Pardoux–Peng's Theorem. Thus, in order to study more general BSDEs, one needs to relax the uniform Lipschitz conditions on the coefficients. In this way, several attempts have been done. Among others, we refer to those with stochastic condition (see El Karoui and Huang, 1997, and Bahlali et al., 2004, 2008).

After they introduced the theory of BSDEs, Pardoux and Peng (1994) in 1994 considered a new kind of BSDEs, that is a class of backward doubly stochastic differential equations (BDSDEs in short) with two different directions of stochastic integrals, i.e., the equations involve both a standard (forward) stochastic integral dW_t and a backward stochastic integral dB_t. They proved the existence and uniqueness of solutions for BDSDEs under uniform Lipschitz conditions on the coefficients.

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More precisely, for any terminal time $T > 0$, under uniform Lipschitz conditions on the coefficients f and g, for any square integrable terminal value ξ, the following BDSDE

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds + \int_t^T g(s, Y_s, Z_s)\overleftarrow{dB_s} - \int_t^T Z_s dW_s, \quad t \in [0, T], \tag{1.2}$$

has a unique solution (Y_t, Z_t) in the interval $[0, T]$. They also showed that BDSDEs can produce a probabilistic representation for solutions to some quasi-linear stochastic partial differential equations (SPDEs). In order to study more general SPDEs, Peng and Shi (2000) have introduced a class of forward–backward doubly stochastic differential equations, under Lipschitz condition on z and monotonicity assumption on y. Many authors have attempted to relax the uniform Lipschitz condition on the coefficients. For instance, several works treat BDSDEs with non-Lipschitz assumptions (see Zhou et al., 2004, Shi et al., 2005, N'zi and Owo, 2009, 2008).

In this paper, we consider BDSDEs with stochastic Lipschitz condition on the coefficients. Doing so, we prove an existence and uniqueness result for fixed terminal time. We also prove a comparison theorem for this class of BDSDEs. Our work provides an extension of result obtained under stochastic Lipschitz condition by El Karoui and Huang (1997) for BSDEs that is when $g \equiv 0$.

The paper is organised as follows. Notations, assumptions and definitions are stated in Section 2. Section 3 deals with the existence and uniqueness. The comparison theorem is treated in Section 4.

2. Notations, assumptions and definitions

2.1. Notations

The standard inner product of $\mathbb{R}^k$ is denoted by $\langle \cdot, \cdot \rangle$ and the Euclidean norm by $|\cdot|$. A norm on $\mathbb{R}^{d\times k}$ is defined by $\sqrt{\mathrm{Tr} ZZ^\star}$, where $Z^\star$ is the transpose of Z. We will also denote this norm by $|\cdot|$.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and T be a fixed final time. Throughout this paper $\{W_t : 0 \le t \le T\}$ and $\{B_t : 0 \le t \le T\}$ will denote two mutually independent standard Brownian motion processes, with values in $\mathbb{R}^d$ and $\mathbb{R}^l$, respectively.

Let $\mathcal{N}$ denote the class of $\mathbb{P}$-null sets of $\mathcal{F}$. For each $t \in [0, T]$, we define

$$\mathcal{F}_t \triangleq \mathcal{F}_t^W \vee \mathcal{F}_{t,T}^B,$$

where for any process $\{\eta_t : t \ge 0\}$; $\mathcal{F}_{s,t}^\eta = \sigma\{\eta_r - \eta_s; s \le r \le t\} \vee \mathcal{N}$ and $\mathcal{F}_t^\eta = \mathcal{F}_{0,t}^\eta$.
Note that $\{\mathcal{F}_{0,t}^W,\ t \in [0, T]\}$ is an increasing filtration and $\{\mathcal{F}_{t,T}^B,\ t \in [0, T]\}$ is a decreasing filtration, and the collection $\{\mathcal{F}_t,\ t \in [0, T]\}$ is neither increasing nor decreasing, so it does not constitute a filtration.

For every random process $(a(t))_{t\ge 0}$ with positive values, such that $a(t)$ is $\mathcal{F}_t$-measurable for any $t \ge 0$, we define an increasing process $(A(t))_{t\ge 0}$ by setting $A(t) = \int_0^t a^2(s)ds$.

For every $\beta > 0$, let $L^2(\beta, a, T, \mathbb{R}^k)$ denote the set of k-dimensional $\mathcal{F}_T$-measurable random variables ξ such that $\|\xi\|_\beta^2 = \mathbb{E}\left(e^{\beta A(T)} |\xi|^2\right) < +\infty$.

Similarly, we denote by $L^2(\beta, a, [0, T], \mathbb{R}^k)$ and $L^{2,a}(\beta, a, [0, T], \mathbb{R}^k)$ the sets of k-dimensional jointly measurable random processes $\{Y_t; t \in [0, T]\}$, such that Y_t is $\mathcal{F}_t$-measurable, for any $t \in [0, T]$ and which satisfy respectively

$$\|Y\|_\beta^2 = \mathbb{E}\left(\int_0^T e^{\beta A(s)} |Y_s|^2 ds\right) < +\infty, \quad \text{and} \quad \|aY\|_\beta^2 = \mathbb{E}\left(\int_0^T e^{\beta A(s)} a^2(s) |Y_s|^2 ds\right) < +\infty.$$

Also, let denote by $L_c^2(\beta, a, [0, T], \mathbb{R}^k)$ the space of k-dimensional continuous random processes $\{Y_t; t \in [0, T]\}$, such that Y_t is $\mathcal{F}_t$-measurable, for any $t \in [0, T]$ and satisfying $\|Y\|_{\beta,c}^2 = \mathbb{E}\left(\sup_{0\le s\le T} e^{\beta A(s)} |Y_s|^2\right) < +\infty$.
Not that the space $L^2(\beta, a, [0, T], \mathbb{R}^k)$ with the norm $\|\cdot\|_\beta$ is a Banach space. So is the space

$$\mathcal{M}(\beta, a, T) = L^{2,a}\left(\beta, a, [0, T], \mathbb{R}^k\right) \times L^2\left(\beta, a, [0, T], \mathbb{R}^{k\times d}\right),$$

with the norm $\|(Y, Z)\|_\beta^2 = \|aY\|_\beta^2 + \|Z\|_\beta^2$. Also is the space

$$\mathcal{M}^c(\beta, a, T) = \left(L^{2,a}\left(\beta, a, [0, T], \mathbb{R}^k\right) \cap L_c^2\left(\beta, a, [0, T], \mathbb{R}^k\right)\right) \times L^2\left(\beta, a, [0, T], \mathbb{R}^{k\times d}\right),$$

with the norm $\|(Y, Z)\|_{\beta,c}^2 = \|Y\|_{\beta,c}^2 + \|aY\|_\beta^2 + \|Z\|_\beta^2$.

Remark 2.1. If $(a(t))_{t\ge 0}$ and $(b(t))_{t\ge 0}$ are two jointly measurable processes with positive values, such that $a(t)$ and $b(t)$ are $\mathcal{F}_t$-measurable for any $t \in [0, T]$, with $b > a$, then $L^2(\beta, b, [0, T], \mathbb{R}^k) \subset L^2(\beta, a, [0, T], \mathbb{R}^k)$. Therefore, $\mathcal{M}^c(\beta, b, T) \subset \mathcal{M}^c(\beta, a, T) \subset \mathcal{M}^c(\beta, 0, T)$.

2.2. Assumptions and definitions

Throughout the paper, the coefficients $f : \Omega \times [0, T] \times \mathbb{R}^k \times \mathbb{R}^{k\times d} \to \mathbb{R}^k$ and $g : \Omega \times [0, T] \times \mathbb{R}^k \times \mathbb{R}^{k\times d} \to \mathbb{R}^{k\times l}$, and the terminal value $\xi : \Omega \to \mathbb{R}^k$ satisfy the following assumptions, for $\beta > 0$: