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A note on global suprema of band-limited spherical random functions

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ABSTRACT

In this note, we investigate the behaviour of suprema for band-limited spherical random fields. We prove upper and lower bound for the expected values of these suprema, by means of metric entropy arguments and discrete approximations; we then exploit the Borell–TIS inequality to establish almost sure upper and lower bounds for their fluctuations. Band limited functions can be viewed as restrictions on the sphere of random polynomials with increasing degrees, and our results show that fluctuations scale as the square root of the logarithm of these degrees.

1. Introduction

The analysis of the behaviour of suprema of Gaussian processes is one of the classical topics in probability theory (Adler and Taylor, 2007; Azaïs and Wschebor, 2009); in this note, we shall be concerned with suprema of band-limited random fields defined on the unit sphere S^2. More precisely, let $T : S^2 \times \Omega \to \mathbb{R}$ be a measurable zero mean, finite variance Gaussian field defined on some probability space $\{\Omega, \Im, P\}$; we assume $T(\cdot)$ is isotropic, e.g. the vectors

$$\{T(x_1), \dots, T(x_k)\} \quad \text{and} \quad \{T(gx_1), \dots, T(gx_k)\}$$

have the same law, for all $k \in \mathbb{N}$, $x_1, \dots, x_k \in S^2$ and $g \in SO(3)$, the group of rotations in $\mathbb{R}^3$. It is then known that the field $\{T(\cdot)\}$ is necessarily mean square continuous (Marinucci and Peccati, 2012) and the following spectral representation holds:

$$T(x) = \sum_{\ell} \sum_{m=-\ell}^{\ell} a_{\ell m} Y_{\ell m}(x),$$

where the *spherical harmonics* $\{Y_{\ell m}\}$ form an orthonormal system of eigenfunctions of the spherical Laplacian, $\Delta_{S^2} Y_{\ell m} = -\ell(\ell+1) Y_{\ell m}$ (see Stein and Weiss, 1971; Marinucci and Peccati, 2011), while the random coefficients $\{a_{\ell m}\}$ form a triangular

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array of complex-valued, zero-mean, uncorrelated Gaussian variables with variance $E\,|a_{\ell m}|^2 = C_\ell$, the *angular power spectrum of the field*. In the sequel, we shall adopt the following general model for the behaviour of $\{C_\ell\}$; as $\ell \to \infty$, there exist $\alpha > 2$ and a positive rational function $G(\ell)$ such that

$$C_\ell = G(\ell)\ell^{-\alpha}, \quad 0 < c_1 < G(\ell) < c_2 < \infty. \tag{1}$$

As a consequence of above, note that $G(\cdot)$ is a smooth function.

Spherical random fields have recently drawn a lot of applied interest, especially in an astrophysical environment (see Bennett et al., 2012; Marinucci and Peccati, 2011); closed form expressions for the density of their maxima and for excursion probabilities have been given in Cheng and Schwartzman (2013), Cheng and Xiao (2012) and Marinucci and Vadlamani (2013). In particular, the latter references exploit the Gaussian Kinematic Fundamental formula by Adler and Taylor (see Adler and Taylor, 2007) to approximate excursion probabilities by means of the expected value of the Euler–Poincarè characteristic for excursion sets. It is then easy to show that

$$\mathbb{E}\mathcal{L}_0(A_u(T)) = 2\left\{1 - \Phi(u)\right\} + 4\pi \left\{\sum_\ell \frac{2\ell+1}{4\pi} C_\ell \frac{\ell(\ell+1)}{2}\right\} \frac{u\phi(u)}{\sqrt{(2\pi)^3}},$$

where $\phi,\ \Phi$ denote density and distribution function of a standard Gaussian variable, while $A_u(T) := \left\{x \in S^2 : T(x) \geq u\right\}$. It is also an easy consequence of results in Ch. 14 of Adler and Taylor (2007) that there exist $\alpha > 1$ and $\mu^+ > 0$ such that, for all $u > \mu^+$

$$\left| \mathbb{P}\left\{\sup_{x \in S^2} T(x) > u\right\} - 2\left\{(1 - \Phi(u)) + u\phi(u)\lambda\right\}\right| \leq \{4\pi\lambda\} \exp\left(-\frac{\alpha u^2}{2}\right), \tag{2}$$

where

$$\lambda := \sum_\ell \frac{2\ell+1}{4\pi} C_\ell \frac{\ell(\ell+1)}{2},$$

denotes the derivative of the covariance function at the origin, see again (Cheng and Schwartzman, 2013; Cheng and Xiao, 2012; Marinucci and Vadlamani, 2013).

When working on compact domains as the sphere, it is often of great interest to focus on sequences of band-limited random fields; for instance, a very powerful tool for data analysis is provided by fields which can be viewed as a sequence of wavelet transforms (at increasing frequencies) of a given isotropic spherical field T. More precisely, take $b(\cdot)$ to be a C^∞ function, compactly supported in $[\frac{1}{2}, 2]$; having in mind the wavelets interpretation, it would be natural to impose the partition of unity property $\sum_\ell b^2(\frac{\ell}{2^j}) \equiv 1$, but this condition however plays no role in our results to follow. Let us now focus on the sequence of band-limited spherical random fields

$$\beta_j(x) := \sum_{\ell=2^{j-1}}^{2^{j+1}} b\left(\frac{\ell}{2^j}\right) \sum_{m=-\ell}^{\ell} a_{\ell m} Y_{\ell m}(x),$$

which have a clear interpretation as wavelet components of the original field, and as such lend themselves to a number of statistical applications, see for instance Baldi et al. (2009a), Cammarota and Marinucci (in press), Narcowich et al. (2006a) and Pietrobon et al. (2008). Band-limited spherical fields have also been widely studied in other contexts of mathematical physics, although in such cases $b(\cdot)$ is not necessarily assumed to be smooth, see for instance Zelditch (2009) and the references therein.

In the sequel, it will be convenient to normalize the variance of $\left\{\beta_j(x)\right\}$ to unity, and thus focus on

$$\widetilde{\beta}_j(x) := \frac{\beta_j(x)}{\sqrt{\sum_\ell b^2\left(\frac{\ell}{2^j}\right)\frac{2\ell+1}{4\pi}C_\ell}}.$$

The sequence of fields $\left\{\widetilde{\beta}_j(x)\right\}$ has covariance functions

$$\rho_j(x,y) = \frac{\sum_\ell b^2\left(\frac{\ell}{2^j}\right)\frac{2\ell+1}{4\pi}C_\ell P_\ell(\langle x,y\rangle)}{\sum_\ell b^2\left(\frac{\ell}{2^j}\right)\frac{2\ell+1}{4\pi}C_\ell}$$

and second spectral moments

$$\lambda_j := \frac{\sum_\ell b^2\left(\frac{\ell}{2^j}\right)\frac{2\ell+1}{4\pi}C_\ell P'_\ell(1)}{\sum_\ell b^2\left(\frac{\ell}{2^j}\right)\frac{2\ell+1}{4\pi}C_\ell} = \frac{\sum_\ell b^2\left(\frac{\ell}{2^j}\right)\frac{2\ell+1}{4\pi}C_\ell\frac{\ell(\ell+1)}{2}}{\sum_\ell b^2\left(\frac{\ell}{2^j}\right)\frac{2\ell+1}{4\pi}C_\ell},$$