Statistics and Probability Letters 96 (2015) 149–153

Contents lists available at ScienceDirect

Statistics and Probability Letters

journal homepage: www.elsevier.com/locate/stapro

Explicit characterization of moments of balanced triangular Pólya urns by an elementary approach

Panpan Zhang *, Chen Chen, Hosam Mahmoud

Department of Statistics, The George Washington University, Washington, DC 20052, USA

ARTICLE INFO

Article history:
Received 2 July 2014
Received in revised form 13 September 2014
Accepted 14 September 2014
Available online 28 September 2014

MSC:
60C05
60F05

Keywords:
Pólya urn
Recurrence
Random structure
Combinatorial probability
Asymptotic analysis

ABSTRACT

Two-color triangular urn models have been investigated recently. Moments of ball counts of a particular color have been obtained exactly and asymptotically in a number of sources (Janson, 2006, Kuba and Panholzer, 2014, and Flajolet et al., 2006). Exact factorial moments are in Kuba and Panholzer (2014). While Flajolet et al. (2006) gives an exact distribution, finding the exact moments from it is daunting. The asymptotic moments in Janson (2006) are derived by heavy machinery—the limit distribution in Janson (2006) is extracted from complicated characteristic functions in the form of integrals that they themselves include functions given by complicated integrals. The methods used in most of these investigations are quite sophisticated. In this note, we characterize the exact moments via an elementary approach. The formulas for the exact moments include Stirling's numbers of second kind and the asymptotic moments follow effortlessly.

1. Introduction

Many evolutionary processes (such as gas diffusion, Ehrenfest and Ehrenfest, 1907, and contagion, Eggenberger and Pólya, 1923) are modeled by the versatile Pólya urn. After the classic work on these models was disseminated, numerous investigations were published on variations and generalizations of the subject. The theory expanded to cover many applications in diverse fields, such as engineering, economics, and experimental and social sciences. The number of papers published on such applications is literally in the thousands, and it is not possible to even list concisely a consistent subset covering all the theoretical works and applications. We only give a reference to the books Johnson and Kotz (1977) and Mahmoud (2008), where hundreds of additional references can be found.

While a theory for many Pólya urn models had been quickly developed, the triangular flavor remained defiant until very recently. It should be noted that the existence of the moments under appropriate scaling is reported by Gouet (1993). The triangular case has been handled in Janson (2006) and limit distributions have been characterized. Also, alternative characterizations are given in Kuba and Mahmoud (in preparation) and Flajolet et al. (2006). The asymptotic characterization in Janson (2006) is derived by heavy machinery. The limiting characteristic function involves integrals of complicated functions that they themselves involve complicated integrals. The exact characterization in Flajolet et al. (2006) does not lend itself easily to exact moment calculations. Such an exact moment calculation involves sums, with a number of summands that steadily increases as the ball drawing continues. The exact factorial moments have very recently computed in Kuba and

* Corresponding author. Tel.: +1 3472858737.
E-mail addresses: pzhang@gwu.edu (P. Zhang), cchen@gwu.edu (C. Chen), hosam@gwu.edu (H. Mahmoud).

http://dx.doi.org/10.1016/j.spl.2014.09.016
0167-7152/

Panholzer (2014), and it is not hard to extract the plain moments from them. However, the first two of these three references use rather sophisticated methods. For instance, Janson (2006) uses branching process and martingales, while Flajolet et al. (2006) uses analytic combinatorics based on systems of differential equations. On the other hand, Kuba and Panholzer (2014) uses a simpler method based on mixed Poisson distributions, and some elements in moment calculations are done in a similar spirit to the present study.

It is our aim in this note to rederive the exact moments by elementary methods, which might be an alternative technique for many similar urn problems. Asymptotic moments follow easily from our presentation.

2. Triangular Pólya urns

A two-color *Pólya urn scheme* is an urn containing balls of two different colors (say white and blue). At each point of discrete time we draw a ball from the urn at random, observe its color and put it back in the urn, and then execute some ball additions according to predesignated rules: if the ball withdrawn is white, we add α white balls and β blue balls; otherwise the ball withdrawn is blue, in which case we add γ white balls and δ blue balls. The four numbers $\alpha, \beta, \gamma, \delta$ are in $\mathbb{N} \cup \{0\}$. More generally, there are other variations where the "additions" are allowed to be negative (i.e., upon withdrawing a color we take balls of some color out of the urn). In these models tenability becomes an issue. We shall not consider urn schemes with ball removal in this note, as they are not in the scope of the urn model to be discussed.

The dynamics of the urn can thus be represented by the *replacement matrix*

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix},$$

in which the rows from top to bottom are indexed by white and blue, and the columns from left to right are also indexed by white and blue; entry (i, j) is the number of balls of color j that we add upon withdrawing a ball of color i.

We consider *balanced urns*, where the number of balls added is a constant, no matter which color appears in the sample. Urn schemes with replacement matrix of the form

$$\begin{pmatrix} \alpha & \beta \\ 0 & \delta \end{pmatrix}$$

are called *triangular*. The case $\alpha = 0$ is degenerate and will be excluded from further study. We also exclude the case $\beta = 0$, as it is the well-studied Eggenberger–Pólya urn; see Eggenberger and Pólya (1923). Developing a theory for balanced triangular urns took a long time after the inception of Pólya urns, let alone triangular schemes that are not balance (and as of yet, there is no complete analysis known for them). We intend to give a characterization by moments of the associated distribution of balanced triangular urn schemes. In such urn models the constant number of balls added after each draw is δ, and necessarily $\beta = \delta - \alpha$.

Let W_n be the number of white balls in a balanced triangular urns after n draws. We characterize the moments of W_n exactly with a formula involving Stirling numbers of the second kind. This characterization is different from those in Kuba and Mahmoud (in preparation).

We also characterize the moments of W_n asymptotically—we shall show that the correct normalization factor is $n^{\alpha/\delta}$, and that $W_n/n^{\alpha/\delta}$ converges in distribution to a random variable uniquely determined by its moments, which involves Gamma functions. This characterization in the limit recovers the result in Flajolet et al. (2006), derived via systems of differential equations; we only employ an elementary induction. The characterization is different from that in Janson (2006).

3. Characterization of the moments

To avoid degeneracy, assume $W_0 \geq 1$. After n draws, the balanced triangular urn contains a total $\tau_n = \delta n + \tau_0$ balls. We only focus on white balls; when we pick a white ball, we add α of them at a time. So, we have a stochastic recurrence for the number of white balls after n draws:

$$W_n = W_{n-1} + \alpha\, \mathbb{I}_n^{(W)}, \tag{1}$$

where $\mathbb{I}_n^{(W)}$ is an indicator of the event of picking a white ball in the nth draw.

The result of this note is in terms of the standard numbers $\left\{ {r \atop j} \right\}$, which are Stirling's numbers of the second kind. These are the number of ways to partition a set of r distinct objects into j nonempty parts. These classic numbers are discussed thoroughly in texts like David and Barton (1962) and Graham et al. (1994).

The proof will be facilitated via the use of Pochhammer's symbols for the falling and rising factorials. The symbol for the falling factorial is

$$(x)_s = x(x-1)\cdots(x-s+1),$$

for any $x \in \mathbb{R}$, and any integer $s \geq 0$, with the interpretation that $(x)_0 = 1$. Pochhammer's symbol for the rising factorial is

$$\langle x \rangle_s = x(x+1)\cdots(x+s-1),$$

for any $x \in \mathbb{R}$, and any integer $s \geq 0$, with the interpretation that $\langle x \rangle_0 = 1$. Note the connection $\langle x \rangle_s = (-1)^s(-x)_s$.

We need the following identity in the proof of the main theorem.