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Counterexamples to a Central Limit Theorem and a Weak Law of Large Numbers for capacities

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ABSTRACT

Chareka (2009) presented two limit theorems in which the additivity requirement of a probability measure is weakened to a one-sided version of the inclusion–exclusion formula. Counterexamples are presented which show that both his Central Limit Theorem and his Weak Law of Large Numbers are false.

1. Introduction

In his paper (Chareka, 2009), Chareka presented a Weak Law of Large Numbers (WLLN) and a Central Limit Theorem (CLT) for variables defined on a capacity space more general than a probability space.

Former results of Marinacci (1999) and others (Maccheroni and Marinacci, 2005; Rébillé, 2009) (see also more recent results like Cozman, 2010; Agahi et al., 2013; Terán, 2014) were surprisingly similar in spirit to the traditional limit theorems, if we take into account that a capacity may fail the cornerstone property of additivity. But, even so, they did not identify a suitable functional which would play the role of the expectation and be the limit in the law of large numbers. Instead, they prove that the upper and lower limits of the sequence of sample means, as the sample size increases, are almost surely confined within a compact interval usually containing more than one point.

Chareka claims to have found a definition of expectation for which those limit theorems admit a presentation with a single limit, mirroring the limit theorems for probability measures. If he were right, that would be a remarkable breakthrough; unfortunately, careful consideration of Chareka's arguments shows that his limit theorems do not hold. We will present two counterexamples substantiating this claim.

A subsidiary objective of this note is to underline that expecting results to hold with no more changes than replacing the word 'probability' by 'capacity' is a flawed heuristics. To that aim, the counterexample to Chareka's WLLN is discussed in detail, showing that the purported extension of the theorem actually contradicts its ordinary probabilistic statement.

2. Preliminaries

Let $(\Omega, \mathcal{A})$ be a measurable space. A *totally monotone capacity* in sense of Chareka (2009), usually called just *a capacity* in the sequel, is a set function $\nu : \mathcal{A} \to [0, 1]$ such that

(i) $\nu(\emptyset) = 0$, $\nu(\Omega) = 1$,
(ii) $A \subset B \Rightarrow \nu(A) \leq \nu(B)$,

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(iii) $A_n \searrow A \Rightarrow \nu(A_n) \to \nu(A)$,
(iv) $A_n \nearrow A \Rightarrow \nu(A_n) \to \nu(A)$,
(v) $\nu(\bigcup_{i=1}^n A_i) \geq \sum_{I\subset\{1,\ldots,n\},I\neq\emptyset}(-1)^{|I|+1}\nu(\bigcap_{i\in I} A_i)$ for any $n \in \mathbb{N}$.

Property (v), *total monotony* (also called complete monotony or infinite monotony), is a one-sided weakening of the inclusion–exclusion formula of probability measures. When working with capacities, the inner continuity condition (iv) is usually relaxed by assuming Ω to be a topological space and demanding it only for open sets instead of all measurable sets. That will make no difference to our argument, though, since both counterexamples satisfy (iv) in the more restrictive sense above.

The complement of an event $A \in \mathcal{A}$ will be denoted by A^c. The indicator function of A will be denoted by I_A.

Events $\{A_i\}_{i\in I}$ are called *independent* (Marinacci, 1999) if

$$\nu\left(\bigcap_{k=1}^n A_{i_k}\right) = \prod_{k=1}^n \nu(A_{i_k})$$

for each $n \in \mathbb{N}$ and each possible choice of a finite subset of different indices $\{i_k\}_{k=1}^n \subset I$. The definition of independent variables follows the same pattern. Although Chareka (2009) fails to specify a definition of independence, he confirmed in a personal communication that the intended meaning was the above.

The *distribution* of a variable X on $(\Omega, \mathcal{A}, \nu)$ is the set function ν_X mapping each Borel set A to $\nu(X^{-1}(A))$. The definition of an *i.i.d.* sequence mirrors the usual one.

Let us explain now the notions of expectation and variance used by Chareka (2009). He considers the *cumulative distribution function* F_X of a variable X in a capacity space (which can also be found e.g. in Denneberg, 1994). Indeed, since the mapping

$$x \mapsto F_X(x) = \nu(X \leq x)$$

is increasing and right-continuous with $\lim_{x\to-\infty} F_X(x) = 0$ and $\lim_{x\to\infty} F_X(x) = 1$, there exists a unique probability measure $P^{(X)}$ on $\mathbb{R}$ such that

$$P^{(X)}((-\infty, x]) = F_X(x) \quad \text{for all } x \in \mathbb{R}.$$

The notation $P^{(X)}$ intentionally sidesteps the usual notation P_X from probability theory, which is potentially misleading here. For instance, $P^{(X+Y)}$ does not relate to $P^{(X)}$ and $P^{(Y)}$ in the same way as for the distributions induced from a probability space. Chareka used the notation μ, which, by omitting the variable, is also misleading as will be shown in the counterexample to the WLLN.

Chareka defines the *expectation* of X to be the expectation of the probability distribution $P^{(X)}$, namely

$$\mathbb{E}[X] = \int_{-\infty}^{\infty} x \mathrm{d}F_X(x).$$

In the special case of the expectation against a probability measure P, we keep the notation E_P.

The *variance* and *standard deviation* of X are then defined as one would expect:

$$\mathbb{D}^2[X] = \mathbb{E}[(X - \mathbb{E}[X])^2], \qquad \mathbb{D}[X] = \sqrt{\mathbb{D}^2[X]}.$$

3. Counterexample to the central limit theorem

We start by taking the sample space $\Omega = \{0, 1\}$, with its algebra of parts and the set function $\nu : \mathcal{P}(\{0, 1\}) \to [0, 1]$ defined by

$$\nu(A) = \begin{cases} 1, & A = \{0, 1\} \\ 0, & A \neq \{0, 1\}. \end{cases}$$

Lemma 3.1. *ν is a totally monotone capacity.*

Proof. This result can be obtained using the Choquet–Matheron–Kendall theorem from random set theory, but a direct proof is short enough.

Properties (i, ii) of a totally monotone capacity are immediate, while properties (iii, iv) are trivial for a finite sample space. As regards (v), there are two possible cases.

Case 1. If $\nu(A_i) = 0$ for all $i = 1, \ldots, n$, then each term in the right-hand side is 0 and so the inequality holds whatever the left-hand side is.

Case 2. If $\nu(A_i) = 1$ for some i, without loss of generality we reorder the A_i to assume that $\nu(A_i) = 1$ for all $i = 1, \ldots, r$, and $\nu(A_i) = 0$ for all $i = r + 1, \ldots, n$. Disposing of $A_{r+1}, \ldots, A_n$ does not change the value of either side of the inequality. Now, all $A_1, \ldots, A_r$ being equal to $\{0, 1\}$, the inequality we must check becomes

$$1 \geq \sum_{I\subset\{1,\ldots,r\},I\neq\emptyset} (-1)^{|I|+1} = \sum_{k=1}^r \binom{r}{k} (-1)^{k+1}. \tag{1}$$