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Large deviations for the boundary local time of doubly reflected Brownian motion☆

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ABSTRACT

We compute a closed-form expression for the moment generating function $\hat{f}(x; \lambda, \alpha) = \frac{1}{\lambda}\mathbb{E}_x(e^{\alpha L_\tau})$, where L_t is the local time at zero for standard Brownian motion with reflecting barriers at 0 and b, and $\tau \sim \text{Exp}(\lambda)$ is independent of W. By analyzing how and where $\hat{f}(x; \cdot, \alpha)$ blows up in λ, a large-time large deviation principle (LDP) for L_t/t is established using a Tauberian result and the Gärtner–Ellis Theorem.

1. Introduction

Diffusion processes with reflecting barriers have found many applications in finance, economics, biology, queueing theory, and electrical engineering. In a financial context, we recall the currency exchange rate target-zone models in Krugman (1991) (see also Svensson, 1991, Bertola and Caballero, 1992, De Jong, 1994, and Ball and Roma, 1998), where the exchange rate is allowed the float within two barriers; asset pricing models with price caps (see Hanson et al., 1999); interest rate models with targeting by the monetary authority (e.g. Farnsworth and Bass, 2003); short rate models with reflection at zero (e.g. Goldstein and Keirstead, 1997, Gorovoi and Linetsky, 2004); and stochastic volatility models (most notably the Heston and Schöbel–Zhu models). In queueing theory, diffusions with reflecting barriers arise as heavy-traffic approximations of queueing systems and reflected Brownian motions is ubiquitous in queueing models (Harrison, 1985; Abate and Whitt, 1987a,b). More recently, reflected Ornstein–Uhlenbeck (OU) and reflected affine processes have been studied as approximations of queueing systems with reneging or balking (Ward and Glynn, 2003a,b). Applications of reflected OU processes in mathematical biology are discussed in Ricciardi and Sacerdote (1987). Doubly reflected Brownian motion also arises naturally in the solution for the optimal trading strategy in the large-time limit for an investor who is permitted to trade a safe and a risky asset under the Black–Scholes model, subject to proportional transaction costs with exponential or power utility (see Guasoni and Muhle-Karbe, in press and Gerhold et al., 2014 respectively).

The asymptotics in this article are obtained using a Tauberian theorem. Tauberian results typically allow us to deduce the large-time or tail behavior of a quantity of interest based on the behavior of its Laplace transform (see Feller, 1971 or the excellent monograph of Bingham et al., 1987 for details or Benaim and Friz, 2008 for applications to tail asymptotics for time-changed exponential Lévy models). In this article, we compute a closed-form expression for the moment generating

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function (mgf) $\hat{f}(x;\lambda,\alpha) = \frac{1}{\lambda}\mathbb{E}_x(e^{\alpha L_\tau})$, where L_t is the local time at zero for standard Brownian motion with reflecting barriers at 0 and b, and τ is an independent exponential random variable with parameter λ. We do this by first deriving the relevant ODE and boundary conditions for $\hat{f}(x;\lambda,\alpha)$ using an augmented filtration and computing the optional projection, and we then solve this ODE in closed form. $\hat{f}(x;\lambda,\alpha)$ does not appear amenable to Laplace inversion; however from an analysis of the location of the pole of $\hat{f}(x;\cdot,\alpha)$, we can compute the re-scaled log mgf limit $V(\alpha) = \lim_{t\to\infty}\frac{1}{t}\log\mathbb{E}_x(e^{\alpha L_t})$ for $\alpha \in \mathbb{R}$ using the Tauberian result in Proposition 4.3 in Korevaar (2002) via the so-called Fejér kernel. From this we then establish a large deviation principle for L_t/t as $t\to\infty$ using the Gärtner–Ellis Theorem from large deviations theory,

Throughout the paper, we let $\mathbb{P}_x(\cdot) = \mathbb{P}(\cdot|X_0 = x)$ denote the law of X given its initial value at time 0 for any $x \in [0, b]$, and by $\mathbb{E}_x(\cdot)$ the expectation under $\mathbb{P}_x$. Further, we let $\mathbb{E} \equiv \mathbb{E}_0$.

2. The modeling set up

We begin by defining the Brownian motion X with two reflecting boundaries. Let W_t be standard Brownian motion starting at 0. Then for any $x \in [0, b]$, there is a unique pair of non-decreasing, continuous adapted processes (L, U), starting at 0, such that

$$X_t = x + W_t + L_t - U_t \in [0, b], \quad \forall t \geq 0$$

such that L can only increase when $X = 0$ and U_t can only increase when $X = b$. Existence and uniqueness follow easily from the more general work of Lions and Sznitman (1984), the earlier work of Skorohod (1962), or a bare-hands proof can be given by successive applications of the standard one-sided reflection mapping using a sequence of stopping times (see Williams, 1992).

It can be shown that

$$\lim_{t\to\infty} L_t/t = \mathbb{E}(L_{\tau^b+\tau'})/\mathbb{E}(\tau^b+\tau'), \qquad \lim_{t\to\infty} U_t/t = \mathbb{E}(U_{\tau^b+\tau'})/\mathbb{E}(\tau^b+\tau'),$$
$$\lim_{t\to\infty}\frac{1}{t}\mathrm{Var}(L_t) = \sigma_L^2, \qquad \lim_{t\to\infty}\frac{1}{t}\mathrm{Var}(U_t) = \sigma_U^2,$$

where $\tau^b = \inf\{t : X_t = b\}$, $\tau' = \inf\{t \geq \tau^b : X_t = 0\}$ (see Williams, 1992) for some non-negative constants σ_L, σ_U.

Proposition 2.1. *Let τ denote an independent exponential random variable with parameter λ. Then for $\alpha < 0$,*

$$\hat{f}(x) \equiv \hat{f}(x;\lambda,\alpha) := \frac{1}{\lambda}\mathbb{E}_x(e^{\alpha L_\tau}) = \int_0^\infty e^{-\lambda t}\,\mathbb{E}_x(e^{\alpha L_t})dt$$

is smooth on $(0, b)$ and satisfies the following ODE

$$\frac{1}{2}\hat{f}_{xx} = \lambda\hat{f} - 1,\ \hat{f}_x(0) + \alpha\hat{f}(0) = \hat{f}_x(b) = 0. \tag{1}$$

Proof. We first show that $\hat{f} \in C^\infty(0, b)$. To this end, note that for $x \in [0, b]$,

$$\mathbb{E}_x(e^{\alpha L_\tau}) = \mathbb{P}_x(\tau > H_0)\,\mathbb{E}_0(e^{\alpha L_\tau}) + \mathbb{P}_x(\tau \leq H_0)$$

where $H_x = \inf\{t : X_t = x\}$ is the first hitting time to x. The law of $(b - X_t; t \in [0, H_0])$ given $X_t = x$ is the same as that of $(|W_t|; t \in [0, H_b])$ given $|W_0| = b - x$. Thus by Eq. 2.0.1 on p. 355 of Borodin and Salminen (2002) we have

$$\mathbb{P}_x(\tau > H_0) = \mathbb{E}_x(e^{-\lambda H_0}) = \frac{\cosh((b-x)\sqrt{2\lambda})}{\cosh(b\sqrt{2\lambda})}.$$

It follows that

$$\mathbb{E}_x(e^{\alpha L_\tau}) = \frac{\cosh((b-x)\sqrt{2\lambda})}{\cosh(b\sqrt{2\lambda})}\,[\mathbb{E}_0(e^{\alpha L_\tau}) - 1] + 1.$$

That is,

$$\hat{f}(x) = \frac{\cosh((b-x)\sqrt{2\lambda})}{\cosh(b\sqrt{2\lambda})}\left(\hat{f}(0) - \frac{1}{\lambda}\right) + \frac{1}{\lambda}, \quad \forall x \in [0, b]. \tag{2}$$

It can then be easily seen from (2) that $\hat{f} \in C^\infty(0, b)$.

To show that $\hat{f}$ satisfies (1) and the boundary conditions, we construct a martingale that is adapted to the filtration generated by X. More specifically, we introduce the natural filtration $\mathcal{F}_t = \sigma(X_s; s \leq t)$ and the augmented filtration