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Limit distributions of generalized St. Petersburg games

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ABSTRACT

The St. Petersburg game is known as a probability model with infinite expectation, and has some extensions. Gut and Martin-Löf (2013) studied convergence in distribution along a suitable specific subsequence under their generalization. In this article, we extend their results using residue analysis investigated by Vardi (1995). We give the explicit limit distributions via characteristic functions with gamma functions which become semi-stable or strictly semi-stable.

1. Introduction

We use notation $\mathbb{N} := \{1, 2, 3, \ldots\}$, $\mathbb{Z} := \{0, \pm 1, \pm 2, \ldots\}$ and $\mathbb{R} := (-\infty, +\infty)$.

1.1. Background and main results

Let us consider that a fair coin is tossed repeatedly until it falls heads. If this happens at the kth trial then a player receives 2^k yen for $k \in \mathbb{N}$. It is known as the *St. Petersburg game*, which has a property that the expectation of the payoff is infinite. This game has been extensively studied (see e.g. Chapter X.4 of Feller, 1968, Section 6.4.1 of Gut, 2013, and Csörgő, 2010), and has several kinds of extension by introducing parameters. Among them Gut and Martin-Löf (2013) generalized a random variable X describing the payoff as follows:

$$\mathrm{P}(X = sr^{k-1}) = pq^{k-1} \quad \text{for } k \in \mathbb{N}, \tag{1}$$

where $s, r > 0$ and $0 < q = 1 - p < 1$. We call the *classical St. Petersburg game* if $s = r = 2$ and $p = q = 1/2$ in (1), which is written in Chapter X.4 of Feller (1968).

Let $\{X_i\}$ be a sequence of independent and identically distributed random variables whose common distribution is given by (1), and set $S_n := \sum_{k=1}^{n} X_k$. If $rq < 1$, then $\mathrm{E}X_1 < \infty$, and as a consequence we apply the classical strong law of large numbers to S_n. Since the expectation of the payoff should be infinite, we suppose that

$$rq \geq 1, \tag{2}$$

namely $\mathrm{E}X_1 = \infty$. Accordingly, the assumption $0 < q < 1$ implies that $r > 1$. Using q and r, we introduce a new parameter

$$\alpha := -\frac{\log q}{\log r}, \tag{3}$$

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where $\log x := \log_e x$ and $\log_r x = \log x / \log r$. Then Eq. (2) yields $0 < \alpha \le 1$, more precisely $\alpha = 1$ and $0 < \alpha < 1$ correspond to $r = 1/q$ and $r > 1/q$, respectively.

For the classical St. Petersburg game, Feller (1968) in Chapter X showed that $\lim_{n\to\infty} S_n/(n \log_2 n) = 1$ in probability. In the setting of (2), Gut and Martin-Löf (2013) gave interesting results including extensions of Feller's weak law of large numbers (see also Gut, 2013, Section 6.4.1).

Now, it follows from $q^{\log x/\log r} = x^{-\alpha}$ that

$$\mathrm{P}(X > x) = q^{1+\lfloor \log(x/s)/\log r \rfloor} = q^{1-\log s/\log r - \langle \log(x/s)/\log r \rangle} x^{-\alpha}, \tag{4}$$

where $\lfloor x \rfloor$ and $\langle x \rangle := x - \lfloor x \rfloor$ denote the integer and the fractional parts of x, respectively. Note that the function of the tail probability $x \mapsto \mathrm{P}(X > x)$ is *not* regularly varying at infinity, because the bounded oscillating coefficient of $x^{-\alpha}$ is not slowly varying. On the other hand, Matsumoto and Nakata (2013) studied the *Feller game* which looks like the St. Petersburg game, however fulfills that the corresponding function of the tail probability is regularly varying. Therefore the Doeblin–Gnedenko theorem (e.g. see Gut, 2013, Theorem 3.2) implies that the distribution is in the domain of attraction of an α-stable distribution. However, in the case of the St. Petersburg game, because of (4) we need to look for other convergence which is not stable.

Györfi and Kevei (2011) studied truncated random variables for the classical St. Petersburg game, and investigated the strong law of large numbers and the central limit theorem by varying truncation levels. In addition, Nakata (in press) extended their results, and investigated a class which can apply their techniques.

Martin-Löf (1985) gave convergence in distribution along a geometric subsequence for the classical St. Petersburg game. Gut and Martin-Löf (2013) directly extended his results. For convenience, let us put

$$\xi(n) := S_n/n - s(1-1/r)\log_r n \quad \text{and} \quad \eta(m,n) := S_m/n. \tag{5}$$

They then obtained as follows:

Theorem 1.1 (*Gut and Martin-Löf, 2013*).

(i) *If* $\alpha = 1$ *then*

$$\xi\left(\lfloor r^n \rfloor\right) \xrightarrow{D} Z_1 \quad \text{as } n \to \infty, \tag{6}$$

where '$\xrightarrow{D}$*' denotes convergence in distribution, and the random variable* Z_1 *is defined via a characteristic function* $\varphi_1(t) := \mathrm{E}\exp(itZ_1)$ *with*

$$\log \varphi_1(t) = \sum_{k=-\infty}^{\infty} \left(\exp(itsr^k) - 1 - itsr^k c_k\right) pq^k \quad \text{for } c_k := \begin{cases} 0, & \text{for } k > 0, \\ 1, & \text{for } k \le 0. \end{cases} \tag{7}$$

(ii) *If* $0 < \alpha < 1$ *then*

$$\eta\left(\lfloor q^{-n} \rfloor, \lfloor r^n \rfloor\right) \xrightarrow{D} Z_2 \quad \text{as } n \to \infty, \tag{8}$$

where the random variable Z_2 *is defined via a characteristic function* $\varphi_2(t) := \mathrm{E}\exp(itZ_2)$ *with*

$$\log \varphi_2(t) = \sum_{k=-\infty}^{\infty} \left(\exp(itsr^k) - 1\right) pq^k. \tag{9}$$

Although Z_1 and Z_2 were actually treated as Lévy processes with continuous parameters in Theorem 2.2 of Gut and Martin-Löf (2013), the easy descriptions are adopted here in order to simplify. In this theorem, it is important that normalized random variables converge in distribution by the specific subsequence. Theorem 1.1 implies that the limit distributions are infinitely divisible, and have Lévy–Khintchine representations (e.g. see Section 8 of Sato (1999)). Moreover, since some calculus with respect to (7) and (9) yields

$$\varphi_1(rt)e^{its(r-1)} = (\varphi_1(t))^r \quad \text{and} \quad \varphi_2(rt) = (\varphi_2(t))^{-q}, \tag{10}$$

it follows from Definition 13.1 of Sato (1999) that the distributions of Z_1 and Z_2 are *semi-stable* and *strictly semi-stable*, respectively. In addition (7) has the form just like Equation 13.3 in Sato (1999). Note that a preliminary result of Theorem 1.1 for the classical St. Petersburg game was proved by Martin-Löf (1985).

On the other hand, Vardi (1995) extended the result of Martin-Löf (1985) from another viewpoint using residue analysis. This method is similar to a technique of a Mellin transformation which associates a function defined on the positive reals. For example, a double exponential sum is expanded for $x \to 0$ as follows (see Example 12 of Flajolet et al., 1995):

$$\sum_{k\ge 0} \exp(-x2^k) = -\log_2 x - \frac{\gamma}{\log 2} + \frac{1}{2} + \frac{1}{\log 2} \sum_{\substack{k\in\mathbb{Z}\\ k\ne 0}} \Gamma\left(-1 + \frac{2k\pi i}{\log 2}\right) e^{-2k\pi i \log_2 x} + \sum_{n=1}^{\infty} \frac{(-x)^n}{n!(1-2^n)},$$