Statistics and Probability Letters 96 (2015) 333–340

Contents lists available at ScienceDirect

Statistics and Probability Letters

journal homepage: www.elsevier.com/locate/stapro

A note on the limiting spectral distribution of a symmetrized auto-cross covariance matrix

Zhidong Bai [a], Chen Wang [b,*]

[a] KLASMOE and School of Mathematics and Statistics, Northeast Normal University, Changchun 130024, People's Republic of China

[b] Department of Statistics and Applied Probability, National University of Singapore, Singapore 117546, Singapore

ARTICLE INFO

Article history:
Received 18 June 2014
Received in revised form 21 August 2014
Accepted 3 October 2014
Available online 26 October 2014

MSC:
primary 60F15
15B52
62H25
secondary 60F05
60F17

Keywords:
Auto-cross covariance
Limiting spectral distribution
Random matrix theory
Stieltjes transform

ABSTRACT

In Jin et al. (2014), the limiting spectral distribution (LSD) of a symmetrized auto-cross covariance matrix is derived using matrix manipulation. The goal of this note is to provide a new method to derive the LSD, which greatly simplifies the derivation in Jin et al. (2014). Moreover, as a by-product, the moment condition of the underlying random variables can be weakened from $2+\delta$ to 2.

© 2014 Elsevier B.V. All rights reserved.

1. Introduction

Consider a large dimensional dynamic k-factor model with lag q taking the form of

$$\mathbf{R}_t = \sum_{i=0}^{q} \mathbf{\Lambda}_i \mathbf{F}_{t-i} + \mathbf{e}_t, \quad t = 1, \ldots, T$$

where $\mathbf{\Lambda}_i$'s are $N \times k$ non-random matrices with full rank. For $t = 1, \ldots, T$, $\mathbf{F}_t$'s are k-dimensional vectors of independent identically distributed (i.i.d.) standard complex components and $\mathbf{e}_t$'s are N-dimensional vectors of i.i.d. complex components with mean zero and finite second moment σ^2, independent of $\mathbf{F}_t$. This can also be considered as a type of *information-plus-noise model* (Dozier and Silverstein, 2007a,b; Bai and Silverstein, 2012) where the information comes from the summation part and the noise is $\mathbf{e}_t$'s. Here both k and q are fixed but unknown, while both N and T tend to ∞ proportionally.

Under this high dimensional setting, an important statistical problem is the estimation of k and q (Bai and Ng, 2002; Harding, submitted for publication). To this objective, the following two variables are defined for fixed non-negative integer

* Corresponding author.
E-mail addresses: baizd@nenu.edu.cn (Z.D. Bai), wangchen2351@gmail.com (C. Wang).

http://dx.doi.org/10.1016/j.spl.2014.10.002
0167-7152/© 2014 Elsevier B.V. All rights reserved.

τ, namely:

$$\mathbf{\Phi}_N(\tau) = \frac{1}{2T}\sum_{j=1}^{T}(\mathbf{R}_j\mathbf{R}_{j+\tau}^* + \mathbf{R}_{j+\tau}\mathbf{R}_j^*)$$

and

$$\mathbf{M}_N(\tau) = \sum_{j=1}^{T}(\gamma_j\gamma_{j+\tau}^* + \gamma_{j+\tau}\gamma_j^*),$$

where $\gamma_j = \frac{1}{\sqrt{2T}}\mathbf{e}_j$ and $*$ denotes the conjugate transpose.

Suppose $\mathbf{A}_n$ is an $n \times n$ random Hermitian matrix with eigenvalues λ_j, $j = 1, 2, \ldots, n$. Define a one-dimensional distribution function

$$F^{\mathbf{A}_n}(x) = \frac{1}{n}\sharp\{j \le n : \lambda_j \le x\}$$

called the empirical spectral distribution (ESD) of matrix A. Here $\sharp E$ denotes the cardinality of the set E. The limit distribution of $\{F^{\mathbf{A}_n}\}$ for a given sequence of random matrices $\{\mathbf{A}_n\}$ is called the limiting spectral distribution (LSD).

Note that when $\tau = 0$, we have $\mathbf{M}_N(\tau) = \frac{1}{T}\sum_{j=1}^{T}\mathbf{e}_j\mathbf{e}_j^*$, which is a sample covariance matrix, whose LSD follows MP law (Marčenko and Pastur, 1967) with density

$$f_c(x) = \frac{1}{2\pi cx}\sqrt{(b_c - x)(x - a_c)}, \quad x \in [a_c, b_c]$$

and a point mass $1 - 1/c$ at the origin if $c > 1$. Here $c = \lim_{N\to\infty} N/T$, $a_c = (1 - \sqrt{c})^2$ and $b_c = (1 + \sqrt{c})^2$.

Moreover, if we write

$$\mathbf{\Lambda} = (\mathbf{\Lambda}_0, \mathbf{\Lambda}_1, \ldots, \mathbf{\Lambda}_q)_{N\times k(q+1)},$$

then the covariance matrix of $\mathbf{R}_t$ will be similar to

$$\begin{pmatrix} \sigma^2\mathbf{I} + \mathbf{\Lambda}^*\mathbf{\Lambda} & \mathbf{0} \\ \mathbf{0} & \sigma^2\mathbf{I} \end{pmatrix},$$

with the size of the upper block and lower block $k(q + 1)$ and $N - k(q + 1)$, respectively. Thus, we have a *spiked population model* (Johnstone, 2001; Baik and Silverstein, 2006; Bai and Yao, 2008). In fact, under certain conditions, the quantity $k(q+1)$ can be estimated by counting the number of eigenvalues of $\mathbf{\Phi}_N(0)$ that are larger than $\sigma^2 b_c$. Therefore, it remains to estimate the numbers k and q separately. To this end, it is necessary to investigate the LSD of $\mathbf{M}_N(\tau)$ for at least one $\tau \ge 1$. As such, Jin et al. (2014) have established the following result.

Theorem 1.1 (*Theorem 1.1 in Jin et al., 2014*). *Assume:*

(a) $\tau \ge 1$ *is a fixed integer.*

(b) $\mathbf{e}_k = (\varepsilon_{1k}, \ldots, \varepsilon_{Nk})'$, $k = 1, 2, \ldots, T + \tau$, *are* N*-dimensional vectors of independent standard complex components with* $\sup_{1\le i\le N, 1\le t\le T+\tau} \mathrm{E}|\varepsilon_{it}|^{2+\delta} \le M < \infty$ *for some* $\delta \in (0, 2]$, *and for any* $\eta > 0$,

$$\frac{1}{\eta^{2+\delta}NT}\sum_{i=1}^{N}\sum_{t=1}^{T+\tau}\mathrm{E}(|\varepsilon_{it}|^{2+\delta}I(|\varepsilon_{it}| \ge \eta T^{1/(2+\delta)})) = o(1). \quad (1.1)$$

(c) $N/(T + \tau) \to c > 0$ *as* $N, T \to \infty$.

(d) $\mathbf{M}_N = \sum_{k=1}^{T}(\gamma_k\gamma_{k+\tau}^* + \gamma_{k+\tau}\gamma_k^*)$, *where* $\gamma_k = \frac{1}{\sqrt{2T}}\mathbf{e}_k$.

Then as $N, T \to \infty$, $F^{\mathbf{M}_N} \xrightarrow{d} F_c$ *a.s. and* F_c *has a density function given by*

$$\phi_c(x) = \frac{1}{2c\pi}\sqrt{\frac{y_0^2}{1 + y_0} - \left(\frac{1 - c}{|x|} + \frac{1}{\sqrt{1 + y_0}}\right)^2}, \quad |x| \le a,$$

where

$$a = \begin{cases} \dfrac{(1 - c)\sqrt{1 + y_1}}{y_1 - 1}, & c \ne 1, \\ 2, & c = 1, \end{cases}$$

y_0 *is the largest real root of the equation:* $y^3 - \frac{(1-c)^2 - x^2}{x^2}y^2 - \frac{4}{x^2}y - \frac{4}{x^2} = 0$ *and* y_1 *is the only real root of the equation:*

$$((1 - c)^2 - 1)y^3 + y^2 + y - 1 = 0 \quad (1.2)$$

such that $y_1 > 1$ *if* $c < 1$ *and* $y_1 \in (0, 1)$ *if* $c > 1$. *Further, if* $c > 1$, *then* F_c *has a point mass* $1 - 1/c$ *at the origin.*

Download English Version:

https://daneshyari.com/en/article/1151597

Download Persian Version:

https://daneshyari.com/article/1151597

Daneshyari.com