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Recursive estimation of time-average variance constants through prewhitening

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ABSTRACT

The time-average variance constant (TAVC) has been an important component in the study of time series. Many real problems request a fast and recursive way in estimating TAVC. In this paper we apply $AR(1)$ prewhitening filter to the recursive algorithm by Wu (2009b), so that the memory complexity of order $O(1)$ is maintained and the accuracy of the estimate is improved. This is justified by both theoretical results and simulation studies.

1. Introduction

The time-average variance constant (TAVC) plays an important role in many time series inference problems such as unit root testing and statistical inference of the mean. Throughout the paper, we focus on a stationary time series $\{X_i\}_{i\in\mathbb{Z}}$ with mean $\mu = E(X_i)$ and finite variance. The TAVC typically has the representation of $\sigma^2 = \sum_{k\in\mathbb{Z}} \gamma(k)$, where $\gamma(k) = \text{cov}(X_0, X_k)$ is the covariance function at lag k. The TAVC is proportional to the spectral density function evaluated at the origin by a constant and the estimation of the latter has been extensively studied in the literature. See for instance Stock (1994), Song and Schmeiser (1995), Phillips and Xiao (1998), Politis et al. (1999), Bühlmann (2002), Lahiri (2003), Alexopoulos and Goldsman (2004) and Jones et al. (2006).

In many applications, one has to sequentially update the estimate of σ^2 while the observations are being accumulated one by one. For the traditional methods of estimating σ^2, the computation time and the memory required will increase for each update of the estimate of σ as the sample size, say n, increases. This is prohibitive especially when multiple MCMC chains are run simultaneously. Wu (2009) modified the batch mean approach into a recursive version without sacrificing the convergence rate of $O(n^{-1/3})$. Meanwhile, the memory complexity of each update is reduced from the traditional $O(n)$ to $O(1)$.

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In this paper, we incorporate the prewhitening idea (Press and Tukey, 1956) into Wu (2009)'s approach in order to achieve reduced mean squared error (MSE) for the estimation of σ^2. The benefit of prewhitening is that the resulting residuals are less dependent than the original data and one can obtain more accurate estimate of the TAVC for the residuals due to larger effective degrees of freedom.[1]

We adopt the $AR(1)$ prefilter in constructing recursive estimate of the TAVC. By this small modification, we would achieve substantial improvements on estimation efficiency under some circumstances. Theoretical conditions are given for deciding if the prewhitening is needed or not. The rest of the paper is organized as follows. Section 2 provides the algorithms of recursive estimation of TAVC. In Section 3, we derive the asymptotic properties of the proposed estimators. In Section 4, we discuss efficiency comparisons between the proposed method and Wu's method in terms of both theoretical results and simulation studies. Section 5 concludes the paper. All the proofs are omitted here and could be found in a longer version of the paper.

2. Proposed algorithms

For notational convenience, we assume $\mu = 0$ throughout the paper without loss of generality. All results in the paper shall remain the same for any other values of μ. We first propose an algorithm for estimating σ^2 through prewhitening when μ is known. Then we generalize it to the unknown case. Here are some frequently used notations throughout the paper: $a_k = \lfloor ck^p \rfloor$; $t_i = \sum_{k\in\mathbb{N}} a_k I_{a_k \le i < a_{k+1}}$; $\rho = \gamma(1)/\gamma(0)$; $\hat{\gamma}_n(k) = n^{-1}\sum_{i=|k|+1}^{n} X_i X_{i-|k|}$; $\tilde{\gamma}_n(k) = n^{-1}\sum_{i=|k|+1}^{n}(X_i - \bar{X}_n)(X_{i-|k|} - \bar{X}_n)$; $\hat{\rho}_n = \hat{\gamma}_n(1)/\hat{\gamma}_n(0)$; $\tilde{\rho}_n = \tilde{\gamma}_n(1)/\tilde{\gamma}_n(0)$; and $v_n = \sum_{i=1}^{n} l_i$, where $l_i = i - t_i + 1$.

2.1. When μ is known

To estimate σ^2, Wu (2009) proposed the statistic $v_n^{-1}\sum_{i=1}^{n}(\sum_{j=t_i}^{i} X_j)^2$, which allows the recursive algorithm with the memory complexity of $O(1)$. One way of improving the accuracy of the estimation is to prewhiten the original sequence so that the transformed sequence has weaker dependence and the corresponding estimation of TAVC would be more accurate. Here, we adopt the $AR(1)$ model as the prefilter to produce a new sequence, i.e. $e_i = X_i - \rho X_{i-1}$ with $\rho = \gamma(1)/\gamma(0)$. Due to the relationship $\sigma^2 = \sigma_e^2/(1-\rho)^2$, it is reasonable to estimate σ^2 by $V_n/(v_n(1-\rho)^2)$ when ρ is known, where

$$V_n = \sum_{i=1}^{n} W_i^2 \quad \text{and} \quad W_i = \sum_{j=t_i}^{i}(X_j - \rho X_{j-1}).$$

When ρ is unknown, one only need to plug in the estimate of ρ and hence σ^2 can be estimated by

$$\hat{\sigma}_n^2 = \hat{V}_n/(v_n(1-\hat{\rho}_n)^2), \tag{1}$$

where $\hat{V}_n = \sum_{i=1}^{n} \hat{W}_{i,n}^2$ and $\hat{W}_{i,n} = \sum_{j=t_i}^{i}(X_j - \hat{\rho}_n X_{j-1})$. It can be shown that

$$\hat{V}_n = \sum_{i=1}^{n}\left(\sum_{j=t_i}^{i} X_j\right)^2 + \hat{\rho}_n^2 \sum_{i=1}^{n}\left(\sum_{j=t_i}^{i} X_{j-1}\right)^2 - 2\hat{\rho}_n \sum_{i=1}^{n}\left(\sum_{j=t_i}^{i} X_j\right)\left(\sum_{j=t_i}^{i} X_{j-1}\right).$$

To simplify notations, let $S_{n,0} = \sum_{i=1}^{n} X_i^2$, $S_{n,1} = \sum_{i=2}^{n} X_i X_{i-1}$, $W_{i,0} = \sum_{j=t_i}^{i} X_j$, $W_{i,1} = \sum_{j=t_i}^{i} X_{j-1}$, $V_{n,0} = \sum_{i=1}^{n} W_{i,0}^2$, $V_{n,1} = \sum_{i=1}^{n} W_{i,1}^2$, and $V_{n,2} = \sum_{i=1}^{n} W_{i,0}W_{i,1}$. As a result, $\hat{\rho}_n = S_{n,1}/S_{n,0}$ and $\hat{V}_n = V_{n,0} + \hat{\rho}_n^2 V_{n,1} - 2\hat{\rho}_n V_{n,2}$. Now $\hat{\sigma}_n^2$ can be calculated recursively by the following Algorithm.

Algorithm 1. At stage n, we store $(k_n, v_n, X_n, S_{n,0}, S_{n,1}, \hat{\rho}_n, W_{n,0}, W_{n,1}, V_{n,0}, V_{n,1}, V_{n,2})$. Note that $t_n = a_{k_n}$. When $n = 1$, the vector is $(1, 1, X_1, X_1^2, 0, 0, X_1, 0, X_1^2, 0, 0)$. At stage $n+1$, we update the vector by

1. If $n+1 = a_{1+k_n}$, let $W_{n+1,0} = X_{n+1}$, $W_{n+1,1} = X_n$ and $k_{n+1} = 1 + k_n$; Otherwise, let $W_{n+1,0} = W_{n,0} + X_{n+1}$, $W_{n+1,1} = W_{n,1} + X_n$ and $k_{n+1} = k_n$.
2. Let $S_{n+1,0} = S_{n,0} + X_{n+1}^2$, $S_{n+1,1} = S_{n,1} + X_{n+1}X_n$, $V_{n+1,0} = V_{n,0} + W_{n+1,0}^2$, $V_{n+1,1} = V_{n,1} + W_{n+1,1}^2$, $V_{n+1,2} = V_{n,2} + W_{n+1,0}W_{n+1,1}$, and $v_{n+1} = v_n + (n + 2 - a_{k_{n+1}})$.
3. Let $\hat{\rho}_{n+1} = S_{n+1,1}/S_{n+1,0}$.
4. Calculate $\hat{V}_{n+1} = V_{n+1,0} + \hat{\rho}_{n+1}^2 V_{n+1,1} - 2\hat{\rho}_{n+1}V_{n+1,2}$

Output: $\hat{\sigma}_{n+1}^2 = \hat{V}_{n+1}/(v_{n+1}(1-\hat{\rho}_{n+1})^2)$.

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[1] For example, suppose the residuals $\{e_i\}_{i\in\mathbb{Z}}$ are generated from $\{X_i\}_{i\in\mathbb{Z}}$ by the mechanism of $a(L)X_i = b(L)e_i$, where L is the lag operate and $a(\cdot)$ and $b(\cdot)$ are polynomial functions so that their roots do not fall in the unit circle. It is well known that $\sigma^2 = a(1)^{-2}b(1)^2\sigma_e^2$, where σ_e^2 is the TAVC of $\{e_i\}_{i\in\mathbb{Z}}$. If $\{X_i\}_{i\in\mathbb{Z}}$ follows an ARMA model, $\{e_i\}_{i\in\mathbb{Z}}$ becomes the white noise sequence under the proper choice of $a(\cdot)$ and $b(\cdot)$. In the latter case, the estimation of σ_e^2 could be a lot more accurate than that of the original sequence.