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Asymptotic efficiency of the OLS estimator with singular limiting sample moment matrices

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ABSTRACT

In the literature on time series analysis, Grenander and Rosenblatt's theorem is necessary to judge the efficiency of OLS estimators with the requirement of Grenander's conditions. However, without the conditions, it is not obvious whether the estimator is efficient. In this study, a model with an asymptotically efficient OLS estimator is presented, regardless whether one of the conditions is not satisfied. The regression model is specified with polynomial regressors of a slowly varying function and general stationary disturbances. The regressors are known to display asymptotic singularity in the sample moment matrices, and thereby, Grenander's condition fails.

1. Introduction

In time-series analysis, discussion of the asymptotic relative efficiency of an ordinary least squares (OLS) estimator dates from the middle of the 20th century. If we denote by $\hat{\beta}_{OLS}$ and $\hat{\beta}_{GLS}$ the OLS and generalized least squares (GLS) estimators of a linear regression model, respectively, the OLS estimator $\hat{\beta}_{OLS}$ is said to be *asymptotically efficient* if

$$\mathrm{Var}(\hat{\beta}_{OLS}) \sim \mathrm{Var}(\hat{\beta}_{GLS}) \tag{1}$$

as the sample size n goes to infinity. One of the most significant contributions to this issue was made by Grenander and Rosenblatt (1957) (GR hereafter). When the regressors are deterministic polynomial functions of time, and the disturbance is stationary, GR have shown the necessary and sufficient condition for (1). In order to judge the efficiency, the GR theorem requires the regressors to satisfy the well-known *Grenander conditions* in the first place. In general, it is not obvious whether (1) is true unless the set of conditions is satisfied and the GR theorem is then applied. For details on the GR theorem and the Grenander conditions, see Anderson (1971, Subsection 10.2.3).

This note reveals the existence of a model whose OLS estimator is asymptotically efficient, even though it does not satisfy one of the Grenander conditions. The result is derived by considering regression with polynomial regressors of a *slowly varying* (SV) function. It is noteworthy that many studies on time series with deterministic trends (e.g., Vogelsang, 1998) employed a general vector of time-trending regressors, like polynomials of time, but ruled out such SV regressors due to their problematic property, notwithstanding their importance in econometrics. In fact, there are surprisingly many examples of

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models with such regressors, including the log-periodogram regression of long memory processes (Robinson, 1995; Hurvich et al., 1998 and references therein), nonlinear least squares estimation of a power exponent (Wu, 1981; Phillips, 2007; Mynbaev, 2011), and research on growth convergence (Barro and Sala-i-Martin, 2004). Focusing on economic convergence and transition modeling, Phillips and Sul (2007, 2009) designed a new model that represents the behavior of economies in transition and proposed an associated test for convergence, utilizing SV functions explicitly.

It is known from Phillips (2007) that SV regressors are asymptotically collinear, and thereby, the Grenander condition fails. Thus, the conventional schemes, including the GR theorem, are not applicable. The basic properties of the regression with SV regressors were investigated first by Phillips (2007). Thereafter, Mynbaev (2009) extended the properties by making some heuristic arguments rigorous. The results up to that point were collected in Mynbaev (2011, Chapter 4). Based on these results, we prove (1) for the regression with SV regressors. The result is interesting as it complements GR's theory.

The remainder of the paper is organized as follows. Section 2 provides some preliminaries and introduces the model. Section 3 states the theorem and Section 4 concludes. Appendix A contains a brief review of the Whittle approximation, which is key to deriving the result. The proof and lemmas are collected in Appendix B.

2. Preliminaries

2.1. Slowly varying function

A positive-valued function L on $[0, \infty)$ is SV if it satisfies, for any $r > 0$, $L(rx)/L(x) \to 1$ as $x \to \infty$. To develop the asymptotic theory, additional conditions are required. Following Mynbaev (2011, Subsection 4.3.1), we determine the class of SV functions by introducing a concept called *remainder*. After this, $\to$ is used when $x \to \infty$.

Definition. We say $L = K(\varepsilon, \phi_\varepsilon)$ if L satisfies all the following conditions.

(a) The function L is SV, and has the *Karamata representation* $L(x) = c_L \exp\left\{\int_B^x s^{-1}\varepsilon(s)ds\right\}$ for $x \geq B$ for some $B > 0$ and $c_L > 0$. The function ε is continuous on $[B, \infty)$, and $\varepsilon(x) \to 0$. (Hereafter, this part is shortened to $L = K(\varepsilon)$.)
(b) The function $|\varepsilon|$ is SV.
(c) There is a positive-valued function ϕ_ε on $[0, \infty)$ called *remainder* such that:
   - ϕ_ε is nondecreasing and $\phi_\varepsilon(x) \to \infty$, and there are constants $\theta, x_0 > 0$ such that $x^{-\theta}\phi_\varepsilon(x)$ is nonincreasing on $[x_0, \infty)$.
   - There is a constant $c > 0$ satisfying $(c\phi_\varepsilon(x))^{-1} \leq |\varepsilon(x)| \leq c\phi_\varepsilon(x)^{-1}$ for $x \geq c$.

Assumption 1. $L = K(\varepsilon, \phi_\varepsilon)$ and $\varepsilon = K(\eta, \phi_\eta)$.

For condition (a), the original Karamata representation allows c_L to be a function of x which converges to a constant, but the simplification, $c_L \equiv$ constant, was made by Phillips (2007). This is justified in the development of asymptotic theory; see Mynbaev (2011, Subsection 4.2.1). Note that L is SV if and only if L has the Karamata representation in the original sense.

Condition (c) was introduced by Mynbaev to make the asymptotic analysis rigorous. For any interval $[a_1, a_2]$ with $0 < a_1 \leq a_2 < \infty$, $L(rx)/L(x) \to 1$ holds uniformly in $r \in [a_1, a_2]$ provided L is SV. However, if r is a function of x and $r \to 0$ at some rate, additional conditions are required. Roughly speaking, condition (c) controls this behavior.

Under assumption $L = K(\varepsilon)$, we obtain

$$\varepsilon(x) = x\varepsilon'(x)/L(x) \to 0. \tag{2}$$

From (2), by further assuming $\varepsilon = K(\eta)$, we observe that for $L_1(x) := \log^\gamma x$ with $\gamma > 0$ and $L_2 := \log\log x$, for example, the corresponding ε- and η-functions are calculated as $\varepsilon_1(x) = \gamma \log^{-1} x$, $\varepsilon_2(x) = (\log x \log\log x)^{-1}$, $\eta_1(x) = -\log^{-1} x$, and $\eta_2(x) = (1 + \log\log x)(\log x \log\log x)^{-1}$. For these practically used SV functions, the remainder is given by $\phi_\varepsilon(x) = |\varepsilon(x)|^{-1}$. In addition, the number $\theta > 0$ can be taken as arbitrarily close to 0 because any SV function is of order $o(x^\theta)$ for any $\theta > 0$. Thus, hereafter, we assume the same (small) $\theta > 0$ for $L = K(\varepsilon, \phi_\varepsilon)$ and $\varepsilon = K(\eta, \phi_\eta)$.

Finally, the assumption $\varepsilon = K(\eta, \phi_\eta)$ implies that L is nondecreasing, which is helpful to streamline the proof clearly. In addition, we can observe that ε is positive, but η is negative and $|\eta|$ becomes SV.

2.2. Regression model

Let L_t denote an SV regressor defined by $L_t = L(t)$, which is supposed to satisfy Assumption 1. We consider the regression model with the polynomials of L_t:

$$y_t = \beta_0 + \beta_1 L_t + \cdots + \beta_p L_t^p + u_t, \quad \text{or} \quad y = X\beta + u, \tag{3}$$

where $y = (y_1, \ldots, y_n)^\top$, $u = (u_1, \ldots, u_n)^\top$, $\beta = (\beta_0, \beta_1, \ldots, \beta_p)^\top$, $X = (\iota, L, \ldots, L^p)$ with $\iota = (1, \ldots, 1)^\top$, and $L^k = (L_1^k, \ldots, L_n^k)^\top$ for $k \in \{1, \ldots, p\}$. In addition, we write $L^0 = \iota$ for convenience. The number p must satisfy the following assumption due to a technical reason.