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A simple probabilistic approach of the Yard-Sale model

Christophe Chorro

Centre d'Economie de la Sorbonne, Maison des Sciences Economiques, 106 bd de l'Hopital, 75013 Paris, France

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ABSTRACT

We propose in the present paper a probabilistic approach of the so-called Hayes (2002) Yard-Sale model using classical diffusion approximations of Markov chains. We partly recover, at the very least for small and high frequency transactions, recent results of Boghosian (2014a) and Boghosian et al. (2015) concerning both wealth condensation in the absence of redistribution mechanisms and steady state distributions when a uniform capital taxation is imposed.

1. Introduction

Several recent studies have questioned the impact of randomness on wealth concentration and proved that luck alone may generate extreme disparities in wealth dynamics even if economical agents are identical in terms of their patience and their abilities (see for example, Fernholz and Fernholz, 2014, Boghosian et al., 2015 and Bouleau and Chorro, 2015). In these models, when there are no redistributive mechanisms, situations of strong concentration appear in which all the wealth ends up in the hands of a single oligarch and taxation is necessary to ensure the existence of a steady state distribution of wealth. Among these approaches, Multiplicative Random Asset Exchange Models (MRAEM), introduced in the framework of Econophysics by Ispolatov et al. (1998), have recently emerged as interesting alternatives to proportional random growth models (see Champernowne, 1953 or Levy, 2005) to describe the evolution of wealth distribution in a population interacting economically. In MRAEM, economy is considered in its simplest form as an interchange of wealth between pairs of agents. This simplicity makes it possible to study social and collective phenomena. In particular, for the so-called Hayes (2002) Yard-Sale model,[1] where a player can win with probability 0.5 a fraction of wealth of the poorer agent, Boghosian et al. (2015) show wealth concentration proving that the associated Gini (1921) coefficient, a classical measure of wealth inequality, is an H function of the Boltzmann equation derived in Boghosian (2014a). In the latter paper, the author also studies the impact of a uniform taxation of capital on wealth dynamics that leads to a steady state distribution for this economic system.

The aim of this paper is to recover these results, at the very least for small and high frequency transactions, using classical probabilistic diffusion approximations in the spirit of Bouleau and Chorro (2015). The main idea is to prove the convergence of the infinitesimal generator associated to the Yard-Sale repeated Markov chain game by transforming the time scales and state spaces appropriately and to study the limiting diffusion process.

For simplicity, we start, in Section 2, with the case of two players where the computations are explicit. The extension to the N players game is discussed in Section 3. Section 4 provides some conclusions.

E-mail address: cchorro@univ-paris1.fr.

[1] As mentioned by Hayes itself in his paper, the transaction mechanism related to the Yard-Sale model has been first introduced in Chakraborti (2002) where the author empirically observed the concentration of wealth after a long time.

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2. The two players case

In the so-called Yard-Sale model, at each round, each player may win or lose, with the same probability, a fraction $a \in [0, 1]$ of the wealth owned by the poorest agent.[2] This model has been introduced in Hayes (2002) to overcome some economic bias of the Theft-and-Fraud model of Ispolatov et al. (1998) where the exchanged amount is a fraction of the losing agent's wealth. In fact, the last model drastically favors the poorest player while the Yard-Sale one is fair in expectation. More precisely, in the Yard-Sale model, if we denote by X_n^i the wealth of player $i \in \{1, 2\}$ after n transactions, supposing that[3] $X_0^1 + X_0^2 = 1$, we have $X_n^1 + X_n^2 = 1$ (zero-sum game) and

$$X_{n+1}^i = X_n^i + a\ \min(X_n^i, 1 - X_n^i)\mathbf{1}_{U_{n+1} \leqslant \frac{1}{2}} - a\ \min(X_n^i, 1 - X_n^i)\mathbf{1}_{U_{n+1} > \frac{1}{2}}$$

where $(U_k)_{k\in\mathbb{N}^*}$ is a sample of the uniform distribution on $[0, 1]$. The sequence $(X_n^i)_{n\in\mathbb{N}}$ is a Markov chain with $\mathbb{E}[X_{n+1}^i \mid X_n^i] = X_n^i$. Thus $(X_n^i)_{n\in\mathbb{N}}$ is a non-negative and bounded martingale that converges almost-surely and in L^p $(1 \leqslant p < \infty)$ toward a random variable X_∞^i. From

$$\mathbb{E}[(X_{n+1}^i - X_n^i)^2 \mid X_n^i] = a^2\ \min(X_n^i, 1 - X_n^i)^2,$$

we deduce that X_∞^i follows a Bernoulli distribution of parameter X_0^i. Thus, even if it is not possible for one player to be ruined after a finite number of rounds, all the wealth is concentrated at the limit in the hands of a single player.

By transforming the time scales and state spaces appropriately it is possible to study the continuous time limit of the preceding Markov chain game. This methodology was used in Bouleau and Chorro (2015) to obtain theoretical approximations of almost-bankruptcy times in elementary market games with proportional bets: Let $f : \mathbb{R} \to \mathbb{R}$ be a measurable and bounded mapping. For all $a \in [0, 1]$ and $x \in]0, 1[$ we define the generator A_a of the elementary market game with parameter a:

$$A_a[f](x) = \mathbb{E}[f(X_1^1) - f(X_0^1) \mid X_0^1 = x] = \frac{1}{2}f(x + a\ \min(x, 1 - x)) + \frac{1}{2}f(x - a\ \min(x, 1 - x)) - f(x).$$

In particular, when f is of class C^∞ with a compact support in the interval $]0, 1[$, we obtain from Taylor expansion that $\frac{1}{a^2}A_a f$ uniformly converges toward $\frac{1}{2}\ \min(x, 1 - x)^2 f''(x)$ when a goes to 0.

Considering the process $(Z_t^a)_{t\in\mathbb{R}_+}$ that is the rescaled (at frequency a^2) continuous time linear interpolation of the sequence $(X_n^1)_{n\in\mathbb{N}}$ with $X_0^1 = x$:

$$\begin{aligned} Z_{na^2}^a &= X_n^1 \quad \forall n \geqslant 0 \\ Z_{(n+\theta)a^2}^a &= Z_{na^2}^a + \theta(Z_{(n+1)a^2}^a - Z_{na^2}^a) \quad \theta \in [0, 1]\ \forall n \geqslant 0, \end{aligned}$$

we obtain from classical arguments (see for example Stroock and Varadhan, 1979, Chap. 11) the uniform weak convergence of $(Z_t^a)_{t\in\mathbb{R}_+}$ when a goes to 0 toward the diffusion process $(X_t)_{t\in\mathbb{R}_+}$, associated to the infinitesimal generator

$$A[f](x) = \frac{1}{2}\min(x, 1 - x)^2 f''(x), \tag{1}$$

that is the unique strong solution of the Stochastic differential equation

$$dX_t = \min(X_t, 1 - X_t)dB_t \quad 0 < X_0 < 1 \tag{2}$$

where B_t is a standard Brownian motion.[4] The points 0 and 1 are absorbing since the constant processes 0 and 1 are solutions of (2). The process $(X_t)_{t\in\mathbb{R}_+}$ is then a continuous and uniformly integrable martingale that converges almost-surely toward 1 with probability X_0 and toward 0 with probability $(1 - X_0)$. Nevertheless, since the Green's function (see Etheridge, 2012, p.44) associated to the diffusion is given by $\forall x \in [0, 1]$,

$$\begin{aligned} G(x, \xi) &= \frac{2x(1 - \xi)}{\min(\xi, 1 - \xi)} \quad \text{if } x < \xi < 1 \\ &= \frac{2\xi(1 - x)}{\min(\xi, 1 - \xi)} \quad \text{if } 0 < \xi < x \end{aligned}$$

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2 The parameter $1 - a$ a may be seen as a measure of the saving propensity supposed to be constant among all the participants to ensure an identical involvement.

3 The zero-sum game hypothesis implies that no wealth is imported, exported, generated, or consumed, wealth can only change hands. Working in a closed economic system, we use in this paper proportions of wealth instead of absolute values.

4 This result of convergence requires the existence and the unicity of the martingale problem associated to the generator (1) that is equivalent to the weak existence and the unicity in distribution of the solution of the associated stochastic differential equation. Here, (2) having Lipschitz we classically deduce the strong existence and unicity of the solution.