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A positive dependence notion based on componentwise unimodality of copulas

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ABSTRACT

A new property defined on the class of symmetric copulas is introduced and studied along this note. It is shown here that such a property can define a family of bivariate distribution functions satisfying all the characteristics listed in Kimeldorf and Sampson (1989) to be considered as a positive dependence notion. Applications, relationships with other positive dependence notions, further properties and the corresponding negative dependence notion are discussed as well.

1. Preliminaries

In the last decades, the concept of positive dependence has played a significant role in many areas of applied probability and statistics, such as reliability or actuarial theory (see, e.g., the monographs by Joe, 1997 and Drouet-Mari and Kotz, 2001, or Lai and Xie, 2006). Starting from the seminal papers by Kimeldorf and Sampson (1987) and Kimeldorf and Sampson (1989), many different notions of positive (or negative) dependence have been studied and applied in the literature to mathematically describe the different aspects and properties of this intuitive concept. Among others, recent theoretical and applied developments in this field are described in Genest and Verret (2002), Colangelo et al. (2005) and Colangelo et al. (2006), Durante et al. (2008a,b), Cai and Wei (2012), Abhishek et al. (2015), and Bignozzi et al. (2015).

Since positive dependence involves several different aspects, Kimeldorf and Sampson in (1987) and Kimeldorf and Sampson in (1989) provided a unified framework for studying and relating three basic concepts of bivariate positive dependence: *positive dependence orderings* (i.e., comparisons based on dependence), *positive dependence notions* (i.e., classes of distributions satisfying some dependence properties) and *measures of positive dependence*. Dealing with positive dependence notions, a general statement for a bivariate property to be considered as a positive dependence notion, or for a class of bivariate distributions to be considered a positive dependence class, has been formulated in Kimeldorf and Sampson (1989) as follows. Here, $\mathcal{F}$ denotes the set of all bivariate distribution functions. Moreover, given any bivariate random vector (X, Y) having joint distribution F, and given $\mathcal{G} \subseteq \mathcal{F}$, the notation $(X, Y) \in \mathcal{G}$ will be sometimes used here and along the paper in place of $F \in \mathcal{G}$, whenever it will simplify the presentation.

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Definition 1.1. A subset $\mathcal{P}^+$ of the family $\mathcal{F}$ is a *positive dependence notion* if it satisfies the following seven conditions.

(C1) $(X, Y) \in \mathcal{P}^+$ implies $P(X > x, Y > y) \geq P(X > x)P(Y > y)$ for all $x, y \in \mathbb{R}$.
(C2) $\mathcal{F}^+ \subseteq \mathcal{P}^+$, where $\mathcal{F}^+$ denotes the set of upper Frèchet bounds, i.e., the set of bivariate distributions F such that $F(x, y) = \min\big(F(x, +\infty), F(+\infty, y)\big)$.
(C3) If (X, Y) is a pair of independent variables, then $(X, Y) \in \mathcal{P}^+$.
(C4) $(X, Y) \in \mathcal{P}^+$ implies $(\phi(X), Y) \in \mathcal{P}^+$ for all increasing functions ϕ.
(C5) $(X, Y) \in \mathcal{P}^+$ implies $(Y, X) \in \mathcal{P}^+$.
(C6) $(X, Y) \in \mathcal{P}^+$ implies $(-X, -Y) \in \mathcal{P}^+$.
(C7) Given the sequence $\{F_n, n \in N\}$, if $F_n \in \mathcal{P}^+$, $\forall n \in N$, and $F_n \to F$ in distribution as $n \to \infty$, then $F \in \mathcal{P}^+$.

It should be pointed out that these seven conditions are logically independent, that is, if any six of them hold, then the seventh need not necessarily hold (see Kimeldorf and Sampson, 1989).

Many of the most well-known positive dependence notions satisfy these axioms. This is for example the case of the *Positive Quadrant Dependence* (PQD) notion, of the *Totally Positive of order 2* (TP_2) notion and of the notion of *Association*, as shown in Kimeldorf and Sampson (1989) and the references therein.

Moreover, since all the monotone dependence properties based on the level of concordance between the components of a random vector (thus, based on rank invariant properties) are entirely described by its copula, whenever it exists and whenever it is unique, all the seven conditions described above can be translated in a more general setting in terms of families of copulas, without taking care of the marginal distributions (except assuming that they are continuous, to ensure a unique representation of the joint distribution with copulas). See, for example, Nelsen et al. (1997) or Fernández Sánchez and Úbeda-Flores (2014) on relationships between copulas and positive dependence notions, indexes and orders. To provide the definition of positive dependence notion in these terms, we recall the definition of copula, and of survival copula.

Let (X, Y) be a random vector with the joint distribution function $F \in \mathcal{F}$ and marginal distributions F_1 and F_2. The function $C : [0, 1]^2 \to [0, 1]$ such that, for all $(x, y) \in \mathbb{R}^2$, satisfies

$$F(x, y) = C(F_1(x), F_2(y))$$

is called *copula* of the vector (X, Y). In this case, it also holds

$$C(u, v) = F(F_1^{-1}(u), F_2^{-1}(v)),$$

for all $u, v \in [0, 1]$, where F_i^{-1} denotes the quasi-inverse of F_i. Such a copula is a bivariate distribution function with margins uniformly distributed on $[0, 1] \subset \mathbb{R}$, and is unique whenever F_1 and F_2 are continuous. For further details on copulas we refer the reader to the standard references (Joe, 1997; Nelsen, 2006). In a similar way is defined the survival copula, which is commonly considered in reliability analysis instead of the copula: given (X, Y) as above, and denoted $\overline{F}, \overline{F}_1$ and $\overline{F}_2$ its joint survival function and the marginal survival functions, its *survival copula* K is defined as

$$K(u, v) = \overline{F}(\overline{F}_1^{-1}(u),\ \overline{F}_2^{-1}(v)) = u + v - 1 + C(1 - u, 1 - v)$$

for all $u, v \in [0, 1]$. See Nelsen (2006) for details.

Using copulas (or survival copulas), and recalling that the copula of a random pair is scale-invariant, thus that (C4) is automatically satisfied, restricting to continuous distribution functions one can restate the conditions described in Definition 1.1 as follows.

Definition 1.2. Let $\mathcal{C}^+$ be a family of bivariate copulas, and let $\mathcal{P}^+$ be the subset of $\mathcal{F}$ of all continuous distribution functions F such that the copula associated with F belongs to $\mathcal{C}^+$. The family $\mathcal{P}^+$ is a *positive dependence notion* if $\mathcal{C}^+$ satisfies the following six conditions.

(C1′) $C(u, v) \geq uv$ for all $(u, v) \in [0, 1]^2$ and for all $C \in \mathcal{C}^+$.
(C2′) The upper Frèchet copula C^+ defined as $C^+(u, v) = \min\big(u, v\big)$ belongs to $\mathcal{C}^+$.
(C3′) The independence copula $C^\perp$ defined as $C^\perp(u, v) = uv$ belongs to $\mathcal{C}^+$.
(C5′) If $C \in \mathcal{C}^+$ then $C^* \in \mathcal{C}^+$, where $C^*(u, v) = C(v, u),\ u, v \in [0, 1]$.
(C6′) If $C \in \mathcal{C}^+$ then for the corresponding survival copula K it holds $K \in \mathcal{C}^+$.
(C7′) Given the sequence $\{C_n, n \in N\}$, if $C_n \in \mathcal{C}^+,\ \forall n \in N$, and $C_n \to C$ in distribution as $n \to \infty$, then $C \in \mathcal{C}^+$.

The motivation of the present study is described now. It is an established fact that the common univariate comparisons among random lifetimes, such as the orders *st* (*usual stochastic order*), *hr* (*hazard rate order*) and *lr* (*likelihood ratio order*), are based on comparisons among the marginal distributions of the involved variables, without taking into consideration their dependence structure. Due to this reason, bivariate characterizations of the most well-known stochastic orders have been defined and studied by several authors, in order to be able to take into account their mutual dependence as well. These characterizations gave rise to new stochastic comparisons, commonly called *joint stochastic orders*, namely, *st:j* (*usual joint stochastic order*), *hr:j* (*joint hazard rate order*) and *lr:j* (*joint likelihood ratio order*), which are equivalent to the original ones under assumption of independence, but are different whenever the variables to be compared are dependent. Also, an alternative weaker version of the joint hazard rate order, namely, *hr:jw* (*weak joint hazard rate order*) has been defined,