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A new class of lifetime distributions

Serkan Eryilmaz

Department of Industrial Engineering, Atilim University, 06836, Incek, Ankara, Turkey

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ABSTRACT

In this paper, a new class of lifetime distributions which is obtained by compounding arbitrary continuous lifetime distribution and discrete phase-type distribution is introduced. In particular, the class of exponential-phase type distributions is studied with some details.

1. Introduction

In the literature, several lifetime distributions were proposed by compounding continuous lifetime distribution with a discrete distribution. The new lifetime distribution mostly arises as the distribution of the random minima which is represented by $\min(X_1, X_2, \ldots, X_N)$, where $X_1, X_2, \ldots$ is a sequence of positive valued independent and identically distributed (i.i.d.) random variables, independent of the discrete random variable N. The most popular choice for the common distribution of the random variables $X_1, X_2, \ldots$ is exponential. Adamidis and Loukas (1998) obtained a new lifetime distribution with decreasing failure rate taking N as a geometric random variable. The resulting distribution has been called exponential geometric distribution. Chahkandi and Ganjali (2009) studied the exponential power series family of distributions, which contains as special cases the exponential geometric, exponential Poisson (Kus, 2007), and exponential logarithmic (Tahmasbi and Rezaei, 2008) distributions. See also Al-Mutairi et al. (2011) and Bourguignon et al. (2014) for the case when N has a power series distribution. Hajebi et al. (2013) proposed an exponential–negative binomial distribution taking N to be a negative binomial random variable. Lu and Shi (2012) studied a new compounding distribution named Weibull–Poisson.

In this paper, we introduce a new class of lifetime distributions by assuming a phase-type distribution for the random variable N. A discrete phase type distribution of order d is defined by considering a Markov chain with d transient states and one absorbing state, say "0". It is the distribution of the time to absorption in an absorbing Markov chain. The probability mass function (pmf) of N which has a discrete phase-type distribution has the form of

$$P\{N = n\} = \mathbf{a}\mathbf{Q}^{n-1}\mathbf{t}', \tag{1}$$

for $n \in \mathbb{N}$, where $\mathbf{Q} = \left(q_{ij}\right)_{d\times d}$ is a matrix which includes the transition probabilities among the d transient states, and $\mathbf{t} = (\mathbf{I} - \mathbf{Q})\mathbf{e}'$ is a vector which includes the transition probabilities from transient states to the absorbing state, $\mathbf{a} = (a_1, \ldots, a_d)$ is the initial probability vector with the entry corresponding to the absorption state removed, $\sum_{i=1}^{d} a_i = 1$, $\mathbf{I}$ is

E-mail address: serkan.eryilmaz@atilim.edu.tr.

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the identity matrix, and $\mathbf{e} = (1, \ldots, 1)_{1\times d}$ (see, e.g. He, 2014). A discrete random variable that has a phase-type distribution will be denoted by $PH_d(\mathbf{a}, \mathbf{Q})$, where d represents the order of the distribution. The matrix $\mathbf{Q}$ must satisfy the condition that $\mathbf{I} - \mathbf{Q}$ is nonsingular. The well-known waiting time distributions such as geometric, negative binomial, and geometric distribution of order k which are defined on a sequence of Bernoulli trials can be represented as a discrete phase-type distribution. Phase type distributions have been widely used in various fields including queueing systems, reliability, and actuarial risk analysis. Their mathematical tractability makes it possible to obtain interesting and useful results. For example, if $N \sim PH_d(\mathbf{a}, \mathbf{Q})$, then the probability generating function of N is given by

$$\phi_N(z) = 1 - \mathbf{a}\mathbf{e}' + \mathbf{a}z\,(\mathbf{I} - z\mathbf{Q})^{-1}\,\mathbf{t}', \tag{2}$$

$0 < z < 1$. For a detailed discussion of phase-type distributions and their properties, we refer to Neuts (1981) and He (2014).

Assuming a phase-type distribution for N is beneficial in several respects. First, as it will be shown in Section 2, it enables us to obtain matrix-based representations for the mixed distributions. Second, some new distribution models can be created for different choices of the main generators $\mathbf{a}$ and $\mathbf{Q}$ of the phase-type random variable N.

The paper is organized as follows. In Section 2, the new class of lifetime distributions is defined by mixing arbitrary distribution F of the sequence $X_1, X_2, \ldots$ and phase type distribution. In Section 3, F is chosen to be exponential and the class of exponential-phase type distributions is studied. In Section 4, we present three special cases of the exponential-phase type distributions when N has a mixed geometric distribution, a negative binomial distribution, and a geometric distribution of order k. Because the case when N follows geometric distribution of order k is first considered in the present paper, in Section 5, we provide extensive numerical results and applications for the exponential–geometric distribution of order k.

2. The class

Let us consider the random variable

$$T = \min(X_1, X_2, \ldots, X_N),$$

where $X_1, X_2, \ldots$ is a sequence of positive valued independent and identically distributed (i.i.d.) random variables, independent of the random variable N.

In the following, we obtain matrix-based representations for the cumulative distribution function (cdf), probability density function (pdf) and hazard rate of the random variable T when the integer valued random variable N has a phase-type distribution with representation $PH_d(\mathbf{a}, \mathbf{Q})$.

Proposition 1. *Let $F(x)$ and $f(x)$ denote respectively the common cdf and pdf of the random variables $X_1, X_2, \ldots$. Then the cdf, pdf and hazard rate of the random variable T are given respectively by*

$$G(x) = \mathbf{a}\mathbf{e}' - \bar{F}(x)\mathbf{a}\left(\mathbf{I} - \bar{F}(x)\mathbf{Q}\right)^{-1}\mathbf{t}', \tag{3}$$

$$g(x) = f(x)\mathbf{a}\left(\mathbf{I} - \bar{F}(x)\mathbf{Q}\right)^{-2}\mathbf{t}', \tag{4}$$

and

$$r(x) = h(x)\frac{\mathbf{a}\left(\mathbf{I} - \bar{F}(x)\mathbf{Q}\right)^{-2}\mathbf{t}'}{\mathbf{a}\left(\mathbf{I} - \bar{F}(x)\mathbf{Q}\right)^{-1}\mathbf{t}'}, \tag{5}$$

for $x \geq 0$, where $h(x) = \frac{f(x)}{\bar{F}(x)}$ is the hazard rate associated with F.

Proof. The survival function of the random variable T can be written as

$$\begin{aligned}\bar{G}(x) &= P\left\{\min(X_1, X_2, \ldots, X_N) > x\right\}\\ &= \sum_n \bar{F}^n(x)P\left\{N = n\right\}\\ &= E(\bar{F}^N(x))\end{aligned}$$

which is, in fact, the probability generating function of N at $\bar{F}(x)$. Thus, from (2)

$$\bar{G}(x) = \phi_N(\bar{F}(x)) = 1 - \mathbf{a}\mathbf{e}' + \mathbf{a}\bar{F}(x)\left(\mathbf{I} - \bar{F}(x)\mathbf{Q}\right)^{-1}\mathbf{t}'.$$

The proofs of Eqs. (4) and (5) are immediate from (3). ■

An alternative representation for the survival function of T is given with the next result.