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Objective priors for the zero-modified model

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ABSTRACT

We consider zero-inflated models, and use the Jeffreys, reference, and matching priors as objective prior distributions derived from the hurdle and with zeros models from the Bayesian viewpoint. We investigate the properties of the resulting posterior distributions.

1. Introduction

Count data are generally the result of recording occurrences and frequencies of events. If count data contain more or less zeros than expected, it is generally called zero-modified data. One example of count data is the number of cups of tea drank in one day. In this case, the data appears to follow a Poisson distribution with more (or less) zeros than expected. We can apply the zero-modified model to such data. When there are more (or less) zeros than expected, we use the zero-inflated (or deflated) model. Count data that has more zeros than expected is called zero-inflated data. Examples of such data include the number of family members with cholera in a village in India (M'Kendrick, 1925), the number of over 80-year-old female deaths per day (Hasselblad, 1969), the number of fetus movements per 5 s (Leroux and Puterman, 1992), the number of HIV infected patients (van den Broek, 1995), and the number of ambulance requests for heat-related illnesses (Bassil et al., 2011). Hall (2000) considered the zero-inflated binomial distribution, Yau et al. (2003) considered the zero-inflated Poisson and negative binomial distributions, and Gupta et al. (1996) and Angers and Biswas (2003) considered the zero-inflated generalized Poisson distribution.

When conducting statistical inferences using an objective Bayesian analysis, we need objective prior distributions. We must provide objective prior distributions that correspond with various situations. Unfortunately, few papers have theoretically studied objective prior distributions for the zero-inflated model. Mullahy (1986) proposed two types of zero-inflated models: the hurdle model and the with-zeros model. Xu et al. (2013) obtained the Jeffreys prior, reference prior, and matching prior for the hurdle Poisson model, and Bhattacharya et al. (2008) obtained the Jeffreys prior for the with-zeros power series model.

The purpose of this article is to obtain major objective priors (the Jeffreys, reference, and matching priors) for general zero-inflated models from the Bayesian perspective. We also consider the properties of the induced posterior distributions. Note that the term "general" means that we do not restrict the underlying distributions of the hurdle or with-zeros.

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2. Two models in the Zero-modified model

We first consider the zero truncated model, which has discrete distributions defined for positive integers. If $f(\cdot|\theta)$ is a probability mass function, then we have a zero-truncated model: $f_T(x|\theta) = f(x|\theta)/(1-f(0|\theta))$, $(x = 1, 2, \ldots)$.

Let $X_1, \ldots, X_n$ be a random sample of size n from the zero truncated model $f_T(x|\theta)$ and let $J(\theta)$ be the Fisher information of $f_T(x|\theta)$. Based on the zero truncated model, Mullahy (1996) defined the "hurdle model" with probability function

$$f_h(x|p,\theta) = \begin{cases} p, & (x=0), \\ (1-p)\dfrac{f(x|\theta)}{1-f(0|\theta)} = (1-p)f_T(x|\theta), & (x = 1, 2, \ldots), \end{cases} \tag{1}$$

where p $(0 \le p \le 1)$ is a mixing rate parameter. The hurdle model assumes that zeros are binomially distributed and positive numbers are from a zero-truncated probability distribution.

The with-zeros model is defined as

$$f_w(x|\omega,\theta) = \begin{cases} \omega + (1-\omega)f(0|\theta), & (x=0), \\ (1-\omega)f(x|\theta), & (x = 1, 2, \ldots). \end{cases} \tag{2}$$

It is possible to tale ω less than zero (van den Broek, 1995). The parameter ω satisfies

$$-f(0|\theta)/(1-f(0|\theta)) \le \omega \le 1. \tag{3}$$

The condition (3) ensure that (2) defines a probability mass function for positive values of ω corresponding to zero-inflated model. If the parameter ω is negative, (2) correspond to zero-deflated model. The with-zeros model has a mixture of distributions at zero and is often used when we test whether $\omega = 0$, i.e. when we test if the data comes from a particular distribution.

Let $\boldsymbol{X}_n = (X_1, \ldots, X_n)$ be a random sample of size n from the zero-inflated model. For an event $A = \{x_i | x_i = 0, i = 1, \ldots, n\}$, we consider the number of the zeros $r_0 = \#\{A\}$, so that $n = r_0 + r_1$ where $r_1 = \#\{A^c\}$. For the above two models, we have the following Fisher informations,

Lemma 1. *For the hurdle model (1), the Fisher information is*

$$H_h(p,\theta) = \begin{pmatrix} \dfrac{1}{p(1-p)} & 0 \\ 0 & (1-p)J(\theta) \end{pmatrix}.$$

For the with zeros model in Eq. (2), the Fisher information is

$$H_w(\omega,\theta) = \begin{pmatrix} \dfrac{1}{(1-\omega)\left(\omega + \frac{f(0|\theta)}{1-f(0|\theta)}\right)} & \dfrac{f'(0|\theta)}{\omega + (1-\omega)f(0|\theta)} \\ \dfrac{f'(0|\theta)}{\omega + (1-\omega)f(0|\theta)} & H_{w22} \end{pmatrix},$$

where H_{w22} in the $(2, 2)$-element is

$$H_{w22} = (1-\omega)\left\{ \frac{\{f'(0|\theta)\}^2}{(1-f(0|\theta))(\omega + (1-\omega)f(0|\theta))} + (1-f(0|\theta))J(\theta) \right\}.$$

3. Objective priors for the two models

The advantage of the objective prior is that we can obtain the same result regardless of our viewpoint. Famous objective priors include the Jeffreys, reference and matching priors.

3.1. Jeffreys prior

The Jeffreys prior is invariant to a one-to-one reparameterization with respect to the parameter (Jeffreys, 1961). Letting $\boldsymbol{\zeta} = (\zeta_1, \ldots, \zeta_k)$ be a parameter vector, the Jeffreys prior $\pi_J(\boldsymbol{\zeta})$ is defined as the square root of the determinant of the Fisher information matrix $\Sigma(\boldsymbol{\zeta})$, i.e. $\pi_J(\boldsymbol{\zeta}) \propto |\Sigma(\boldsymbol{\zeta})|^{1/2}$, where $|\Sigma(\boldsymbol{\zeta})|$ is the determinant of the matrix $\Sigma(\boldsymbol{\theta})$. The Jeffreys prior for the hurdle model is $\pi_{hJ}(p,\theta) \propto |H_h(p,\theta)|^{1/2} = p^{-1/2}|J(\theta)|^{1/2}$, and the Jeffreys prior for the with zeros model is

$$\begin{aligned} \pi_{wJ}(\omega,\theta) &\propto |H_w(\omega,\theta)|^{1/2} \\ &= \left(\omega + \frac{f(0|\theta)}{1-f(0|\theta)}\right)^{-1/2} (1-f(0|\theta))^{1/2}|J(\theta)|^{1/2}. \end{aligned}$$