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Tail probabilities of solutions to a generalized Ait-Sahalia interest rate model

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ABSTRACT

In this paper we consider a generalized Ait-Sahalia interest rate model. We first extend the space of admissible parameters that ensures the existence of a unique positive solution to the model. Then, we provide an explicit estimate for tail probabilities of solutions.

1. Introduction

The original Ait-Sahalia interest rate model was proposed first by Ait-Sahalia (1996) and then, has been investigated by various authors (see, e.g. Conley et al., 1997, Gallant and Tauchen, 1997, Hong and Haitao, 2005). The aim of this paper is to provide some new contributions to a generalized Ait-Sahalia model that was recently introduced by Szpruch et al. (2011). In this model, the dynamics of interest rates is described by the following stochastic differential equation

$$dx_t = \left(\frac{a_1}{x_t} - a_2 + a_3 x_t - a_4 x_t^r\right)dt + \sigma x_t^\rho dw_t, \tag{1.1}$$

where w_t is a standard Brownian motion, the coefficients $a_1, \ldots, a_4, \sigma$ and the initial condition x_0 are positive constants.

The drift and the volatility of the model (1.1) violate the Lipschitz and linear growth conditions which are traditionally imposed in the study of stochastic differential equations. In addition, the drift has a singularity at $x = 0$. Those cause some mathematical difficulties which make the study of the model (1.1) particularly interesting. As a mathematical model arising in finance, there are a lot of fundamental properties of the system (1.1) that need discussing. When $\rho > 1$ and $r > 1$, the existence, uniqueness and numerical simulation of positive solutions have been investigated by Szpruch et al. (2011). In the present paper, we provide the following results:

(i) In Theorem 3.1, we show that the model (1.1) admits a unique positive solution even when $\rho \in (0, 1)$ and $r \in (-1, 1)$. To obtain a such result, our proof strongly relies on the singularity of the drift, the Hölder continuity of Brownian motion and the technique of stopping times. We also notice that the technique used in Szpruch et al. (2011) is different from ours and cannot be applied to the model (1.1) with $\rho \in (0, 1)$ and $r \in (-1, 1)$.

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(ii) Because the probability distribution function is one of the most natural features for any random variable, it would be desirable to study the probability distribution of solutions to (1.1). In fact, the probability distribution of the solution to several interest rate models can be computed explicitly. For example, the solution of Cox–Ingersoll–Ross model follows a non-central Chi-Squared distribution (Cox et al., 1985), the probability distribution of the solution to Jacobi model can be represented by a series of Jacobi polynomials (Delbaen and Shirakawa, 2002). However, the exact distribution of the solution x_t to (1.1) is unknown, it is still an open problem. Hence, one would like to obtain an estimate for probability distributions (the estimates for the CIR/CEV-type models can be found in De Marco, 2011). Based on the results established recently in Dung et al. (2015) and Nourdin and Viens (2009), we will provide an explicit estimate for probability distributions of x_t in Theorem 3.2.

The rest of the paper is organized as follows. In Section 2, we recall some fundamental concepts of Malliavin calculus which will be a main tool to prove Theorem 3.2. The main results of the paper are stated and proved in Section 3.

2. Preliminaries

Let us recall some elements of stochastic calculus of variations (for more details see Nualart, 2006). Fix a time interval $[0, T]$. We suppose that $(w_t)_{t\in[0,T]}$ is defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$, where $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$ is a natural filter generated by the Brownian motion w. For $h \in L^2[0, T]$, we denote by $w(h)$ the Wiener integral: $w(h) = \int_0^T h(t)dw_t$. Let $\mathcal{S}$ denote the dense subset of $L^2(\Omega, \mathcal{F}, P)$ consisting of smooth random variables of the form

$$F = f(w(h_1), \dots, w(h_n)), \tag{2.1}$$

where $n \in \mathbb{N}, f \in C_b^\infty(\mathbb{R}^n), h_1, \dots, h_n \in L^2[0, T]$. If F has the form (2.1), we define its Malliavin derivative as the process $DF := \{D_tF, t \in [0, T]\}$ given by

$$D_tF = \sum_{k=1}^{n} \frac{\partial f}{\partial x_k}(w(h_1), \dots, w(h_n))h_k(t).$$

We shall denote by $\mathbb{D}^{1,2}$ the space of Malliavin differentiable random variables, it is the closure of $\mathcal{S}$ with respect to the norm

$$\|F\|_{1,2}^2 := E|F|^2 + \int_0^T E|D_uF|^2 du.$$

Based on the results from Dung et al. (2015), Nourdin and Viens (2009), we have the following estimate for tail probabilities of a Malliavin differentiable random variable.

Proposition 2.1. *Let F be in $\mathbb{D}^{1,2}$ with mean zero. We define the following function in $\mathbb{R}$:*

$$\varphi_F(x) := E\left[\int_0^\infty D_tF\, E[D_tF \mid \mathcal{F}_t]\, dt \,\Big|\, F = x\right].$$

Assume that $0 < \varphi_F(F) \le \alpha F + \beta$, a.s. for some $\alpha \ge 0$ and $\beta > 0$. Then, for all $z > 0$, we have

$$P(F \ge z) \le \exp\left(-\frac{z^2}{2\alpha z + 2\beta}\right) \quad \text{and} \quad P(F \le -z) \le \exp\left(-\frac{z^2}{2\beta}\right). \tag{2.2}$$

Proof. Follows directly from Theorem 4.1 in Nourdin and Viens (2009) and Proposition 2.3 in Dung et al. (2015). □

3. The main results

We will not directly do the proof for the model (1.1), but for a coordinate transformation thereof. Put $y_t = x_t^{1-\rho}$, by Itô's formula we get

$$dy_t = (1-\rho)\left(\frac{a_1}{x_t^{1+\rho}} - a_2x_t^{-\rho} + a_3x_t^{1-\rho} - a_4x_t^{r-\rho} - \frac{1}{2}\sigma^2\rho x_t^{-1+\rho}\right)dt + (1-\rho)\sigma dw_t,$$

or equivalently

$$dy_t = (1-\rho)\left(\frac{a_1}{y_t^{\frac{1+\rho}{1-\rho}}} - \frac{a_2}{y_t^{\frac{\rho}{1-\rho}}} + a_3y_t - \frac{a_4}{y_t^{\frac{\rho-r}{1-\rho}}} - \frac{\sigma^2\rho}{2y_t}\right)dt + (1-\rho)\sigma dw_t, \quad t \ge 0. \tag{3.1}$$

Thus this transformation allows us to shift the nonlinearity from the volatility coefficient into the drift coefficient. Then the results can be more easily proved.