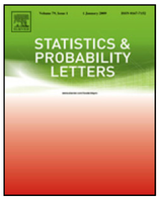
Statistics and Probability Letters 112 (2016) 131–136

Contents lists available at ScienceDirect

Statistics and Probability Letters

journal homepage: www.elsevier.com/locate/stapro



Stochastic comparisons of parallel and series systems with heterogeneous Birnbaum–Saunders components

Longxiang Fang[a,b], Xiaojun Zhu[b,*], N. Balakrishnan[b]

[a] *Department of Mathematics and Computer Science, Anhui Normal University, Wuhu 241000, PR China*
[b] *Department of Mathematics and Statistics, McMaster University, Hamilton, Ontario, Canada L8S 4K1*

ARTICLE INFO

Article history:
Received 6 May 2015
Received in revised form 27 January 2016
Accepted 27 January 2016
Available online 10 February 2016

MSC:
60E15
62G30
62D05
62N05

Keywords:
Birnbaum–Saunders distribution
Log Birnbaum–Saunders distribution
Usual stochastic order
Parallel systems
Series systems
Majorization

ABSTRACT

In this paper, we discuss stochastic comparisons of lifetimes of parallel and series systems with independent heterogeneous Birnbaum–Saunders components with respect to the usual stochastic order based on vector majorization of parameters. Specifically, let $X_1, \ldots, X_n$ be independent random variables with $X_i \sim BS(\alpha_i, \beta_i), i = 1, \ldots, n$, and $X_1^*, \ldots, X_n^*$ be another set of independent random variables with $X_i^* \sim BS(\alpha_i^*, \beta_i^*), i = 1, \ldots, n$. Then, we first show that when $\alpha_1 = \cdots = \alpha_n = \alpha_1^* = \cdots = \alpha_n^*$, $(\beta_1, \ldots, \beta_n) \succeq_m (\beta_1^*, \ldots, \beta_n^*)$ implies $X_{n:n} \geq_{st} X_{n:n}^*$ and $(\frac{1}{\beta_1}, \ldots, \frac{1}{\beta_n}) \succeq_m (\frac{1}{\beta_1^*}, \ldots, \frac{1}{\beta_n^*})$ implies $X_{1:n}^* \geq_{st} X_{1:n}$. We subsequently generalize these results to a wider range of the scale parameters. Next, we show that when $\beta_1 = \cdots = \beta_n = \beta_1^* = \cdots = \beta_n^*$, $(\frac{1}{\alpha_1}, \ldots, \frac{1}{\alpha_n}) \succeq_m (\frac{1}{\alpha_1^*}, \ldots, \frac{1}{\alpha_n^*})$ implies $X_{n:n} \geq_{st} X_{n:n}^*$ and $X_{1:n}^* \geq_{st} X_{1:n}$. Finally, we establish similar results for the log Birnbaum–Saunders distribution.

1. Introduction

Let $X_{1:n} \leq \cdots \leq X_{n:n}$ denote the order statistics from random variables $X_1, \ldots, X_n$. These quantities play an important role in many areas including statistical inference, operations research, reliability theory, life testing, and quality control. Interested readers may refer to the volumes by Balakrishnan and Rao (1998a,b) for relevant details. Evidently, in reliability theory, $X_{k:n}$ is the lifetime of a $(n-k+1)$-out-of-n system, which is a popular structure of redundancy in fault-tolerant systems. In particular, $X_{n:n}$ and $X_{1:n}$ correspond to the lifetimes of parallel and series systems. Stochastic comparisons of parallel and series systems with heterogeneous components have been studied by many authors including Dykstra et al. (1997), Khaledi and Kochar (2000), Khaledi and Kochar (2006), Balakrishnan and Zhao (2013), Li and Li (2015), Balakrishnan et al. (2015), and Torrado and Kochar (2015).

Birnbaum and Saunders (1969a,b) introduced a two-parameter failure time distribution for fatigue failure caused under cyclic loading. This distribution has been used quite effectively for modeling positively skewed data, especially for data on

* Corresponding author.
E-mail addresses: lxfang@fudan.edu.cn (L. Fang), xiaojunzhu8899@hotmail.com (X. Zhu), bala@mcmaster.ca (N. Balakrishnan).

http://dx.doi.org/10.1016/j.spl.2016.01.021

crack growth and life-times. A random variable X is said to have the Birnbaum–Saunders distribution ("BS" in short) if its cumulative distribution function (cdf) is given by

$$F(x; \alpha, \beta) = \Phi\left[\frac{1}{\alpha}\left(\sqrt{\frac{x}{\beta}} - \sqrt{\frac{\beta}{x}}\right)\right], \quad x > 0,$$

where $\Phi(\cdot)$ denotes the standard normal cdf, and $\alpha > 0$ and $\beta > 0$ are the shape and scale parameters, respectively. In this case, we denote it by $X \sim BS(\alpha, \beta)$.

Many authors have studied different aspects of this model and for some recent work, one may refer to Chang and Tang (1993); Dupuis and Mills (1994), Díaz-García and Leiva-Sánchez (2005), Ng et al. (2003), Ng et al. (2006), Kundu et al. (2008), Leiva et al. (2008), Sanhueza et al. (2008), Zhu and Balakrishnan (2015), and the references cited therein. For a brief survey on BS distribution and its properties, see Johnson et al. (1995). Furthermore, Rieck and Nedelman (1991) introduced log Birnbaum–Saunders distribution and showed that it can be obtained as a special case of the sinh-normal distribution. This distribution has many interesting properties in addition to being useful as a log-linear model for lifetime data. A random variable Y is said to have a log Birnbaum–Saunders distribution ("LBS" in short) if its cdf is given by

$$F(y; \alpha, \beta) = \Phi\left[\frac{2}{\alpha}\sinh\left(\frac{y - \ln\beta}{2}\right)\right], \quad y \in R,$$

where $\Phi(\cdot)$ is the standard normal cdf as before, and $\sinh(y)$ is the hyperbolic sine function defined as $\sinh(y) = \frac{e^y - e^{-y}}{2}$. We denote it by $Y \sim LBS(\alpha, \beta)$. It is clear that

$$X \sim BS(\alpha, \beta) \Longleftrightarrow \ln X \sim LBS(\alpha, \beta). \tag{1.1}$$

In this paper, we study the lifetimes of parallel and series systems with independent BS and LBS components with respect to the usual stochastic order based on vector majorization of parameters. Specifically, let $X_1, \ldots, X_n$ be independent random variables with $X_i \sim BS(\alpha_i, \beta_i)$ (or $LBS(\alpha_i, \beta_i)$), $i = 1, \ldots, n$, and $X_1^*, \ldots, X_n^*$ be another set of independent random variables with $X_i^* \sim BS(\alpha_i^*, \beta_i^*)$ (or $LBS(\alpha_i^*, \beta_i^*)$), $i = 1, \ldots, n$. Then, we first show that when $\alpha_1 = \cdots = \alpha_n = \alpha_1^* = \cdots = \alpha_n^*$,

$$(\beta_1, \ldots, \beta_n) \succeq_m (\beta_1^*, \ldots, \beta_n^*) \Longrightarrow X_{n:n} \ge_{st} X_{n:n}^*$$

and

$$\left(\frac{1}{\beta_1}, \ldots, \frac{1}{\beta_n}\right) \succeq_m \left(\frac{1}{\beta_1^*}, \ldots, \frac{1}{\beta_n^*}\right) \Longrightarrow X_{1:n}^* \ge_{st} X_{1:n}.$$

We subsequently generalize these results to a wider range of the scale parameters as follows:

$$(\beta_1, \ldots, \beta_n) \succeq_w (\beta_1^*, \ldots, \beta_n^*) \Longrightarrow X_{n:n} \ge_{st} X_{n:n}^*$$

and

$$\left(\frac{1}{\beta_1}, \ldots, \frac{1}{\beta_n}\right) \succeq_w \left(\frac{1}{\beta_1^*}, \ldots, \frac{1}{\beta_n^*}\right) \Longrightarrow X_{1:n}^* \ge_{st} X_{1:n}.$$

Finally, we establish that when $\beta_1 = \cdots = \beta_n = \beta_1^* = \cdots = \beta_n^*$,

$$\left(\frac{1}{\alpha_1}, \ldots, \frac{1}{\alpha_n}\right) \succeq_m \left(\frac{1}{\alpha_1^*}, \ldots, \frac{1}{\alpha_n^*}\right) \Longrightarrow X_{n:n} \ge_{st} X_{n:n}^* \quad \text{and} \quad X_{1:n}^* \ge_{st} X_{1:n}.$$

Since BS distribution has become one of the popular lifetime models in reliability literature and that we discuss stochastic orderings for largest and smallest order statistics, the results established directly relate to some key distributional properties and features of parallel and series systems, two most common coherent systems, with BS components. Furthermore, these results may also be useful in establishing some statistical properties of estimators of the scale and shape parameters of the BS distribution. We are currently looking into this problem and hope to report the findings in a future paper.

Throughout this paper, "increasing" and "decreasing" are used to mean "nondecreasing" and "nonincreasing", respectively.

2. Preliminaries

In this section, we provide a brief review of definitions of usual stochastic order, majorization and weak majorization. For more details, one may refer to Shaked and Shanthikumar (2007) and Marshall et al. (2011).

Definition 2.1. We say that Y is smaller than X in the usual stochastic order, denoted by $X \ge_{st} Y$, if $\bar{F}(x) \ge \bar{G}(x)$ for all x.