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When the bispectrum is real-valued

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ABSTRACT

Let $\{X(t), t \in \mathbb{Z}\}$ be a stationary time series with a.e. positive spectrum. Two consequences of that the bispectrum of $\{X(t), t \in \mathbb{Z}\}$ is real-valued but nonzero are: (1) if $\{X(t), t \in \mathbb{Z}\}$ is also linear, then it is reversible; (2) $\{X(t), t \in \mathbb{Z}\}$ cannot be causal linear. A corollary of the first statement: if $\{X(t), t \in \mathbb{Z}\}$ is linear, and the skewness of $X(0)$ is nonzero, then third order reversibility implies reversibility. In this paper the notion of bispectrum is of a broader scope since we do not assume the absolute summability of the third order cumulants.

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1. Introduction

If a time series is reversible, then all of its polyspectra, if they exist, are real-valued. The frequency-domain test of reversibility in Hinich and Rothman (1998) uses this property regarded to the bispectrum, i.e. a necessary condition of reversibility is that the bispectrum is real valued. In this paper we prove essentially that when the time series is linear, the real-valuedness of the non-zero bispectrum is a sufficient condition as well, see Theorem 1. This confirms that when linearity is known to hold, then for testing reversibility (1) there is no need for polyspectra of order higher than three, and (2) the bispectrum-based reversibility test of Hinich and Rothman (1998) is consistent (with respect to non-reversibility, and not only with respect to the alternative hypothesis that the bispectrum is not real). There is also another corollary, valid essentially for linear time series with a skewed distribution: third order reversibility (see Definition 2) implies reversibility, see Corollary 1.

Our other theorem states basically: if the spectrum is positive and the bispectrum is real-valued but nonzero, then the time series cannot be causal linear, see Theorem 2.

Let us recall some notions. A time series $\{X(t), t \in \mathbb{Z}\}$ is called *reversible*, if $(X_t, X_{t+1}, \ldots, X_{t+k}) \stackrel{d}{=} (X_{t+k}, X_{t+k-1}, \ldots, X_t)$ for all $k \in \mathbb{N}$ and $t \in \mathbb{Z}$ ($\stackrel{d}{=}$ means equality in distribution). Reversibility implies stationarity, see Lawrance (1991). A time series $\{X(t), t \in \mathbb{Z}\}$ is reversible, if and only if it is stationary and

$$(X_{t_1}, \ldots, X_{t_k}) \stackrel{d}{=} (X_{-t_1}, \ldots, X_{-t_k}) \tag{1}$$

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for all $k \in \mathbb{N}$ and $t_1 < \cdots < t_k \in \mathbb{Z}$. Since Gaussian stationary time series are always reversible, it is enough to deal with the non-Gaussian case.

A time series $\{X(t), t \in \mathbb{Z}\}$ is called *linear*, if it has a moving average representation

$$X(t) = \sum_{k=-\infty}^{\infty} c(k)Z(t-k), \tag{2}$$

with i.i.d. random variables $Z(t),\ t \in \mathbb{Z}$, $\mathrm{E}Z(t) = 0$, $\mathrm{E}Z(t)^2 < \infty$ and $\sum_{k=-\infty}^{\infty} c(k)^2 < \infty$. Obviously, linearity implies stationarity. Because of non-Gaussianity, the representation (2) is unique, apart from constant multiplier and time shift, see Cheng (1992) or Rosenblatt (2000, Theorem 1.3.1). A linear representation of the type (2) is called *causal*, if the summation is over nonnegative indices only, i.e. if the time series does not depend on future $Z(t)$ values.

The proofs of the main results depend largely on the solution of a particular case of the Cauchy functional equation, see Lemma 4.

The rest of the paper is organized as follows. In Section 2 the notion of the bispectrum is generalized in order to be existing for, among others, linear time series with finite third order moment. The main results of the paper are stated in Section 3. The proofs and the necessary lemmas on the Cauchy functional equation are presented in Section 4.

2. The bispectrum of a linear time series

Assume that the time series $\{X(t), t \in \mathbb{Z}\}$ is stationary in third order. Let us denote the joint cumulant of the random variables $X(t_1), X(t_2), X(t_3)$ by $\mathrm{cum}\,(X(t_1), X(t_2), X(t_3))$. Because of stationarity we have $\mathrm{cum}\,(X(t_1), X(t_2), X(t_3)) = \mathrm{cum}\,(X(0), X(t_2-t_1), X(t_3-t_1))$, $t_1, t_2, t_3 \in \mathbb{Z}$, thus the third order joint cumulant is, in fact, a function of two variables only. The bispectrum is defined usually as the two variable Fourier-transform of the sequence of third order joint cumulants $\mathrm{cum}\,(X(0), X(t_1), X(t_2))$, $t_1, t_2 \in \mathbb{Z}$, Brillinger (1981). For the Fourier-transform to be meaningful, it has been required that the cumulant series be absolutely summable. Defined in this way, the bispectrum is an integrable function, thus $\mathrm{cum}\,(X(0), X(t_1), X(t_2))$, $t_1, t_2 \in \mathbb{Z}$, is the two variable inverse Fourier-transform of it. The absolute summability condition is, however, too strict, e.g. long range dependent time series generally fail to fulfil it. However, if we define the bispectrum requiring the integrability of the bispectrum only, but not the absolute summability of the cumulant series, then we get a more general concept, which is extensive enough to apply to at least any linear time series with finite moments of third order, see Brillinger (1965) and Terdik (2011). On the other hand, the integrability of a function guarantees the one-to-one correspondence between itself and its inverse Fourier transform. Thus, the above mentioned usual definition of the bispectrum can be generalized so that we do not assume absolutely summable cumulants.

Definition 1. Let $\{X(t), t \in \mathbb{Z}\}$ be a stationary time series with finite third order moment, and assume that a function $B(\omega_1, \omega_2)$ defined a.e. on $[0, 2\pi) \times [0, 2\pi)$ is integrable, and its inverse Fourier transform is just the cumulant sequence, i.e.

$$\mathrm{cum}\,(X(0), X(t_1), X(t_2)) = \int_0^{2\pi} \int_0^{2\pi} \exp\left(i\,(t_1\omega_1 + t_2\omega_2)\right) B(\omega_1, \omega_2) d\omega_1 d\omega_2, \tag{3}$$

$t_1, t_2 \in \mathbb{Z}$. Then the function $B(\omega_1, \omega_2)$ is called the bispectrum of $\{X(t), t \in \mathbb{Z}\}$.

In the rest of the paper we use the notion of bispectrum in the sense of Definition 1. As we have already mentioned, there is a one-to-one correspondence between the set of all bispectra and the set of those third order joint cumulant sequences for which the bispectrum exists.

Lemma 1. *Let $\{X(t), t \in \mathbb{Z}\}$ be a linear time series with finite third order moment and moving average representation (2). Then the bispectrum of $\{X(t), t \in \mathbb{Z}\}$ exists, and it has the form*

$$B(\omega_1, \omega_2) = \left(\mathrm{cum}_3(Z(0))/(2\pi)^2\right) \varphi(\omega_1)\varphi(\omega_2)\varphi(-\omega_1 - \omega_2) \tag{4}$$

for a.e. $(\omega_1, \omega_2) \in [0, 2\pi) \times [0, 2\pi)$, where

$$\varphi(\omega) = \sum_{k=-\infty}^{\infty} c(k)e^{-ik\omega},$$

$\omega \in [0, 2\pi)$, is the frequency domain transfer function corresponding to the linear representation (2).

3. New relations among reversibility, bispectrum and linearity

Theorem 1. *Let $\{X(t), t \in \mathbb{Z}\}$ be a linear time series with finite third order moment and a.e. positive spectrum. If its bispectrum is real-valued but not a.e. zero, then $\{X(t), t \in \mathbb{Z}\}$ is reversible.*

Sometimes the 2 and 3 dimensional distributions can be handled more directly than the general finite dimensional ones. Motivated by this, we introduce the following notion, and then state a corollary of the previous theorem.

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