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Empirical processes of iterated maps that contract on average

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ABSTRACT

We consider a Markov chain obtained by random iterations of Lipschitz maps T_i chosen with a probability $p_i(x)$ depending on the current position x. We assume this system has a property of "contraction on average", that is

$$\sum_i d(T_i x, T_i y) p_i(x) < \rho d(x, y)$$

for some $\rho < 1$. In the present note, we study the weak convergence of the empirical process associated to this Markov chain.

1. Introduction

The study of the limit behavior of partial sums of random variables has a long time interest in dynamical systems or Markov chains theory. Existence of (attractive) invariant measure, law of large numbers, central limit theorem, almost sure invariance principle are results that characterize the statistical properties of such systems. In this note, we study the limit behavior of some random iterative Lipschitz models which are introduced in the following section. Several limit theorems for partial sums are already known for such models (see Peigné, 1993, Walkden, 2007). The purpose of this note is to obtain limit theorems for the associated empirical processes. The weak convergence of empirical processes is also of interest for dynamical systems in order to derive statistical tests, as Kolmogorov–Smirnov test. See for example Collet et al. (2004) where expanding maps of the interval are considered.

The iterative Lipschitz models that we study are described in Sections 2 and 3. Some background on empirical processes is briefly recalled in Section 4. In Sections 5 and 6, we show how results of Dehling et al. (in press) can be applied to get an empirical central limit theorem for the iterative Lipschitz models considered here.

2. Iterated Lipschitz maps

Let $\mathcal{X}$ be a locally compact metric space, with a countable basis. Denote by d the metric on $\mathcal{X}$. Let T_i, $i \geq 0$, be a sequence of Lipschitz maps from $\mathcal{X}$ to $\mathcal{X}$ and p_i, $i \geq 0$, be a sequence of Lipschitz functions from $\mathcal{X}$ to $[0, 1]$ such that for all $x \in \mathcal{X}$, $\sum_{i \geq 0} p_i(x) = 1$. We will consider the Markov chain with state space $\mathcal{X}$ and transition probability P given by

$$P(x, A) = \sum_{i \geq 0} p_i(x) 1_A(T_i x), \tag{1}$$

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for all $x \in \mathcal{X}$ and all Borel subset A of $\mathcal{X}$. We also denote by P the Markov operator defined for measurable functions f on $\mathcal{X}$ by

$$Pf(x) = \int_{\mathcal{X}} f(y)P(x, dy) = \sum_{i\geq 0} p_i(x)f(T_i x).$$

We assume the maps T_i contract on average, that is, there exists $\rho \in (0, 1)$ such that for all $x, y \in \mathcal{X}$,

$$\sum_{i\geq 0} d(T_i x, T_i y)p_i(x) < \rho d(x, y). \tag{2}$$

The case of constant p_i has been studied by Dubins and Freedman (1966) and Barnsley and Elton (1988) (with applications to image encoding) or Hennion and Hervé (2001). In particular, an empirical CLT has been proved by Wu and Shao (2004). For variable p_i, such systems have been considered by Doeblin and Fortet (1937) and Karlin (1953) (with applications in learning models), Barnsley et al. (1988) (existence of invariant measure), Peigné (1993) (CLT), Pollicott (2001) (Berry–Essen bounds) or Walkden (2007) (AISP). A lot of concrete examples, that we do not present in this short note, can be found in the articles cited above.

For our general setting, we need the following extra assumptions that are the assumptions H0, H1, H2, and H4 of Peigné (1993). For some fixed $x_0 \in \mathcal{X}$, we assume that

$$\sup_{x,y,z\in\mathcal{X}, y\neq z} \sum_{i\geq 0} \frac{d(T_i y, T_i z)}{d(y, z)} p_i(x) < +\infty, \tag{3}$$

$$\sup_{x,y\in\mathcal{X}} \sum_{i\geq 0} \frac{d(T_i y, x_0)}{1 + d(y, x_0)} p_i(x) < +\infty, \tag{4}$$

$$\sup_{x\in\mathcal{X}} \sum_{i\geq 0} \frac{d(T_i x, x_0)}{1 + d(x, x_0)} \sup_{y,z\in\mathcal{X}, y\neq z} \frac{|p_i(y) - p_i(z)|}{d(y, z)} < +\infty. \tag{5}$$

Remark that these three conditions are trivially satisfied when the family of maps T_i is finite.

We will also assume that, for any $x, y \in \mathcal{X}$, there exist sequences of integer $(i_n)_{n\geq 1}$ and $(j_n)_{n\geq 1}$ such that, for some $x_0 \in \mathcal{X}$,

$$d(T_{i_n} \circ \cdots \circ T_{i_1} x, T_{j_n} \circ \cdots \circ T_{j_1} y)(1 + d(T_{j_n} \circ \cdots \circ T_{j_1} x, x_0)) \xrightarrow[n\to+\infty]{} 0, \tag{6}$$

with $p_{i_n}(T_{i_{n-1}} \circ \cdots \circ T_{i_1} x) \dots p_{i_1}(x) > 0$ and $p_{j_n}(T_{j_{n-1}} \circ \cdots \circ T_{j_1} y) \dots p_{i_1}(x) > 0$ for all $n \geq 1$. Remark that this assumption is satisfied when (2)–(5) hold and the p_i are all strictly positive.

Also observe that our condition (2), instead of the condition H3 of Peigné (1993), is sufficient to obtain Theorem 1 below. Peigné (1993) consider systems that contract on average before k_0 steps. For simplicity, we only consider here the case $k_0 = 1$ but our result remains valid in Peigné's setting. See Peigné (1993) for further comments on these assumptions.

3. Spaces of Hölder's functions with weights

We will introduce some Banach spaces, on which the operator P acts with good spectral properties. Let α and β be non-negative real numbers. We denote by $\mathcal{H}_{\alpha,\beta}$ the space of continuous functions from $\mathcal{X}$ to $\mathbb{R}$ such that $\|f\|_{\alpha,\beta} = N_\beta(f) + m_{\alpha,\beta}(f) < +\infty$, where

$$N_\beta(f) = \sup_{x\in\mathcal{X}} \frac{|f(x)|}{1 + d(x, x_0)^\beta} \quad \text{and} \quad m_{\alpha,\beta}(f) = \sup_{x,y\in\mathcal{X}, x\neq y} \frac{|f(x) - f(y)|}{d(x, y)^\alpha(1 + d(x, x_0)^\beta)}.$$

$\|\cdot\|_{\alpha,\beta}$ is a norm on $\mathcal{H}_{\alpha,\beta}$ and the space $\mathcal{H}_{\alpha,\beta}$, equipped with this norm, is a Banach space. We will also use the notation $\|f\|_\infty = \sup_{x\in\mathcal{X}} |f(x)|$. The following statement is easily verifiable.

Lemma 1. *If $f, g \in \mathcal{H}_{\alpha,\beta}$ with $\|f\|_\infty < +\infty$ and $\|g\|_\infty < +\infty$, then $fg \in \mathcal{H}_{\alpha,\beta}$ and*

$$\|fg\|_{\alpha,\beta} \leq \|g\|_\infty \|f\|_{\alpha,\beta} + \|f\|_\infty \|g\|_{\alpha,\beta}.$$

The interest of introducing these Banach spaces is the following spectral decomposition of the Markov operator which is due to Peigné (1993).

Theorem 1 (*Peigné, 1993*)**.** *Assume (2)–(6) hold. Let $\alpha, \beta \in (0, 1/2)$ with $\alpha < \beta$. Then*

(1) *P operates on $\mathcal{H}_{\alpha,\beta}$.*
(2) *There exists an attractive P-invariant probability measure ν which has a finite first moment, i.e., $\int_{\mathcal{X}} d(x, x_0)\nu(dx) < \infty$.*
(3) *There exists a bounded operator Q with spectral radius strictly less than 1 such that $P = \nu + Q$, with $\nu Q = Q\nu = 0$.*

Here, ν is an attractive probability measure means that for all probability measure μ, $P^n\mu$ converges to ν in distribution.