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Asymptotics for the residual-based bootstrap approximation in nearly nonstationary AR(1) models with possibly heavy-tailed innovations[☆]

Ke-Ang Fu[a,∗], Yuechao Li[a], Andrew Cheuk-Yin Ng[b]

[a] School of Statistics and Mathematics, Zhejiang Gongshang University, Hangzhou 310018, China

[b] Department of Finance, The Chinese University of Hong Kong, Shatin, N.T., Hong Kong

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ABSTRACT

Consider a nearly nonstationary AR(1) model, $X_t = \theta_n X_{t-1} + u_t$, where $\theta_n = 1 - \gamma/n$, γ is a fixed constant, and the innovations are in the domain of attraction of the normal law with possibly infinite variance. As for the least squares estimator $\hat{\theta}_n$ of θ_n, we propose to use a residual-based m-out-of-n bootstrap procedure to approximate the distribution of $\hat{\theta}_n - \theta_n$, and its asymptotic validity is proved.

1. Introduction

Consider the time series $\{X_t, t \geq 1\}$ given by the model

$$X_t = \theta_n X_{t-1} + u_t, \quad X_0 = 0, \tag{1.1}$$

where X_t is the observation at time t, $\theta_n = 1 - \gamma/n$ with γ being a fixed real number and $\{u_t\}$ is random disturbance. Model (1.1) is known as the nearly nonstationary AR(1) model, and when $\gamma = 0$, model (1.1) reduces to unit root AR(1) model. Given the observations $X_0, X_1, \ldots, X_n$, θ_n is customarily estimated by its least-squares estimator (LSE):

$$\hat{\theta}_n = \frac{\sum_{i=1}^{n} X_{i-1} X_i}{\sum_{i=1}^{n} X_{i-1}^2}. \tag{1.2}$$

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∗ Corresponding author.
E-mail addresses: fukeang@hotmail.com (K.-A. Fu), lucas1101@163.com (Y. Li), andrewng@baf.msmail.cuhk.edu.hk (A.C.-Y. Ng).

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As for this LSE, Chan and Wei (1987) and Phillips (1987) have studied the asymptotic distribution with $\{u_t\}$ being martingale difference and strong mixing sequences, respectively. When θ_n is close to one, due to the lack of smooth transition, the asymptotic distribution of $\hat{\theta}_n$ does not converge to a standard normal variable, but a functional of Brownian motions. Particularly, Chan and Wei (1987) provided the following theorem.

Theorem A. *As for the model (1.1), let $\{u_t\}$ be a martingale difference sequence with respect to an increasing sequence of σ-fields $\{\mathcal{F}_t\}$ such that as $n \to \infty$,*

$$\frac{1}{n}\sum_{t=1}^{n} \mathrm{E}(u_t^2|\mathcal{F}_{t-1}) \xrightarrow{p} 1$$

and for any $\alpha > 0$,

$$\frac{1}{n}\sum_{t=1}^{n} \mathrm{E}(u_t^2 I(|u_t| > \alpha\sqrt{n})|\mathcal{F}_{t-1}) \xrightarrow{p} 0.$$

Then, as $n \to \infty$,

$$\sqrt{\sum_{t=1}^{n} X_{t-1}^2}(\hat{\theta}_n - \theta_n) \xrightarrow{d} \mathcal{L}(\gamma),$$

where $\xrightarrow{d}$ ($\xrightarrow{p}$) means convergence in distribution (probability),

$$\mathcal{L}(\gamma) := \frac{\int_0^1 (1+bt)^{-1} W(t) dW(t)}{\sqrt{\int_0^1 (1+bt)^{-2} W^2(t) dt}}$$

with $b = e^{2\gamma} - 1$, and $\{W(t), t \geq 0\}$ is a standard Wiener process. In particular, when $\gamma = 0$, Itô's formula implies $\mathcal{L}(0) = \frac{W^2(1)-1}{2\sqrt{\int_0^1 W^2(t)dt}}$.

It is easy to see the i.i.d. case is a special case of Theorem A. Later, the nearly nonstationary AR(1) models were investigated by many statisticians with various assumptions on $\{u_t\}$. For instance, if $\{u_t\}$ is i.i.d. and in the domain of attraction of a strictly α-stable law with $\alpha \in (0, 2)$, Chan (1990) derived the limiting distribution of the LSE. Buchmann and Chan (2007) and Chan and Zhang (2009) gave the asymptotic theory of the LSE under long range dependence with finite variance and infinite variance, respectively. It is worth to mention that the nearly nonstationary AR(1) processes were generalized to AR(p) processes by Jeganathan (1991, 1999), and the asymptotic behaviors of the LSE were also discussed in the articles.

Recently, Hwang and Pang (2009) established the limiting distribution of the LSE, when $\{u_t\}$ is i.i.d. and in the domain of attraction of the normal law (DAN) with possibly infinite variance, and got the following result.

Theorem B. *In model (1.1), if $\{u_t\}$ is i.i.d. and belongs to the DAN with mean zeros and possibly infinite variances, then as $n \to \infty$,*

$$n(\hat{\theta}_n - \theta_n) \xrightarrow{d} \frac{W^2(B_\gamma(1)) + 2\gamma\int_0^1 e^{2\gamma(1-t)} W^2(B_\gamma(t))dt - 1}{2\int_0^1 e^{2\gamma(1-t)} W^2(B_\gamma(t))dt} =: \zeta(\gamma),$$

and

$$\sqrt{\frac{n\sum_{t=1}^{n} X_{t-1}^2}{\sum_{t=1}^{n}(X_t - \theta_n X_{t-1})^2}}(\hat{\theta}_n - \theta_n) \xrightarrow{d} \mathcal{L}(\gamma), \tag{1.3}$$

where $B_\gamma(t) = e^{-2\gamma}(e^{2t\gamma} - 1)/(2\gamma)$.

If $\{u_t\}$ is an i.i.d. sequence with mean zeros and finite second moments, then it is easy to see $\sum_{t=1}^{n}(X_t - \theta_n X_{t-1})^2/n \xrightarrow{p} \mathrm{E}u_1^2$, which guarantees that (1.3) coincides with Theorem A under i.i.d. assumption.

It is well known that the great interest of getting the asymptotic distribution is the opportunity of building confidence intervals for the unknown autoregressive parameter. However, it is a heavy task to compute the bias and variance of the random variable involved in Theorems A and B. One way to overcome this problem is to use a bootstrap procedure. Hence, our interest of this paper is to propose a bootstrap methodology, and show the LSE could be approximated by using residual bootstrap method.

This paper is organized as follows. In the next section, we first propose the residual-based m-out-of-n bootstrap process, and then show the asymptotic distribution of the bootstrap statistic. The proofs of the asymptotic result are collected in Section 3.