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A recursive pricing formula for a path-dependent option under the constant elasticity of variance diffusion

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ABSTRACT

In this paper, we consider a path-dependent option in finance under the constant elasticity of variance diffusion. We use a perturbation argument and the probabilistic representation (the Feynman–Kac theorem) of a partial differential equation to obtain a complete asymptotic expansion of the option price in a recursive manner based on the Black–Scholes formula and prove rigorously the existence of the expansion with a convergence error.

1. Introduction

In finance, it is well-known that Cox (1975) and Cox and Ross (1976) initiated the constant elasticity of variance (in brief, CEV) diffusion model as an extension of the celebrated Black–Scholes model (Black and Scholes, 1973) to overcome some disadvantages of the model. One major limit of the Black–Scholes model is that the option price based on it would not create the desirable non-flat geometry, usually called "smile", of implied volatilities. Graphing implied volatilities against strike prices for a given expiry lead to a skewed smile instead of the "flat" surface. In this sense, the CEV model has become one of the successful underlying models representing the dynamics of assets such as stock, and index. Representing the underlying risky asset price by X_t with a given expiry T, the CEV model is given by the stochastic differential equation (SDE)

$$\begin{aligned} dX_t &= \mu X_t dt + \sigma X_t^{\frac{\theta}{2}} dW_t, \quad 0 < t \le T, \\ X_0 &= x \end{aligned} \tag{1}$$

on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a filtration $(\mathcal{F}_t)_{t\in[0,T]}$ generated by one-dimensional Brownian motion W_t. Here, θ, μ and σ are given positive constants. The particular case $\theta = 2$ corresponds to the Black–Scholes model.

It is well-known that the transition density function of the CEV diffusion X_t consists of an infinite sum of the Bessel functions (cf. Lipton (2001), Schroder (1989)) so that computing the price of options based on this model heavily relies on numerical calculations in many cases. So, the CEV model is very hard to be applied to many practical problems for exotic options which are coped with in over-the-counter markets even if there is an analytic formula for the option prices. This motivates us to choose a path-dependent option, i.e., lookback option, under the CEV model and find a formula which can be more easily computed in practice. In fact, the analytic pricing formula for barrier and lookback options under the CEV model

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has been obtained by Davydov and Linetsky (2001). However, the transformation technique used there requires a rather difficult numerical computation for the option price. On the contrary, our formula to be obtained in this paper is given by a recursive form that can be calculated starting from the well-known Black–Scholes formula and so computing the option price is relatively easy. The idea of this approach is based upon the observation that many underlying risky assets turn out to follow approximately the log-normal distribution as noted by Park and Kim (2011). That is, the elasticity parameter θ should be not exactly 2 but close to 2, which will provide us with analytic tractability as well as practical advantage in the pricing of derivatives.

Among many exotic path-dependent options, lookback option is a type of option leading to a mathematical challenge for computing the no-arbitrage price. The payoff structure of lookback option depends on the maximum or minimum value of the underlying asset's price occurring over the life of the option. The option holder needs to observe the underlying until expiry date. There are two kinds of lookback option, i.e., one with floating strike and one with fixed strike. The case of floating strike type has been studied by Park and Kim (2011) using the Green function method. Then the price of fixed type lookback option can be given by the well-known (model-free) fixed–floating parity relation on them. From the view point of mathematical approach, however, the price of fixed type lookback option is hard to be obtained directly by the Green function method. So, this paper develops a more universal method mentioned above which can be applied to not only the price of lookback option with fixed strike under the CEV model but also other option pricing problems where the Green function method may not be available. Also, we provide a rigorous mathematical proof of convergence on the asymptotic pricing formula.

The structure of this paper is organized as follows. In Section 2, we establish a partial differential equation (PDE) problem for the price of lookback call option with fixed strike. In Section 3, we obtain a lemma which is an analogue of the well-known Feynman–Kac theorem (Oksendal (2003)) and find an asymptotic expansion of the option price under the assumption that the elasticity parameter of the CEV diffusion is close to 2 corresponding to the Black–Scholes model. In Section 4, we estimate rigorously the error of the asymptotic formula to show the existence of the expansion. Finally, Section 5 concludes.

2. Option pricing

We first assume that there is a martingale (risk neutral) probability measure $\mathbb{Q}$ such that the SDE (1) can be transformed into

$$\begin{aligned} dX_t &= rX_t dt + \sigma X_t^{\frac{\theta}{2}} d\widetilde{W}_t, \quad 0 < t \le T, \\ X_0 &= x, \end{aligned} \tag{2}$$

where $\widetilde{W}_t$ is a Brownian motion under the measure $\mathbb{Q}$ and r is a positive constant (interest rate).

To define the lookback option price, we use the maximal and minimal processes defined by

$$\begin{aligned} X_t^\star &= \sup_{s\le t} X_s, \\ X_t^- &= \inf_{s\le t} X_s, \end{aligned}$$

respectively. In fact, the supremum and infimum here can be replaced by the maximum and minimum, respectively, since the process X_t is continuous in t almost surely. Then the risk neutral price of general lookback option is defined by

$$P(t,x,x^\star,x^-) = E^{\mathbb{Q}}[e^{-r(T-t)}H(X_T,X_T^\star,X_T^-)|X_t = x, X_t^\star = x^\star, X_t^- = x^-], \tag{3}$$

where H is called a payoff function. The payoff function has two types, i.e.,

$$\begin{aligned} H^{fl} &= X^\star - X_T, \\ H^{fi} &= (X^\star - K)^+. \end{aligned}$$

The option with payoff H^{fl} is called lookback put option with floating strike and the option with H^{fi} lookback call option with fixed strike, respectively. As stated in the introduction, this paper is concerned with lookback option with fixed strike.

We want to obtain an option pricing problem satisfied by a lookback call option with fixed strike in a PDE form. In order to do it, we use the same argument as in Chapter 7 of Shreve (2000) and Park and Kim (2013) as follows. First, we note that the solution of the SDE (2) is continuous and so is $X_t^\star$. Moreover, $X_t^\star$ is nondecreasing. Then, from the definition of quadratic covariance process denoted by $[X, Y]_t$, we observe that

$$[X,X^\star]_t = \lim_{|\pi|\to 0}\sum_{i=0}^{n}(X_{t_{i+1}} - X_{t_i})(X^\star_{t_{i+1}} - X^\star_{t_i}) \le X_t^\star \lim_{|\pi|\to 0}\sup |X_{t_{i+1}} - X_{t_i}| = 0 \quad \text{a.s.}$$

$$[X^\star,X^\star]_t = \lim_{|\pi|\to 0}\sum_{i=0}^{n}|X^\star_{t_{i+1}} - X^\star_{t_i}|^2 \le X_t^\star \lim_{|\pi|\to 0}\sup |X^\star_{t_{i+1}} - X^\star_{t_i}| = 0 \quad \text{a.s.}$$