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On infinitely divisible distributions with polynomially decaying characteristic functions

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ABSTRACT

We provide necessary and sufficient conditions on the characteristics of an infinitely divisible distribution under which its characteristic function φ decays polynomially. Under a mild regularity condition this polynomial decay is equivalent to $1/\varphi$ being a Fourier multiplier on Besov spaces.

1. Introduction

Any infinitely divisible distribution (IDD) μ is uniquely determined by its characteristic triplet (σ^2, γ, ν) where $\sigma^2 > 0$ is the diffusion coefficient, $\gamma \in \mathbb{R}$ is a drift parameter and ν is the so-called Lévy measure. By the Lévy–Khintchine formula the characteristic function of μ is given by

$$\begin{aligned}\varphi(u) : &= \mathcal{F}\mu(u) := \int e^{iux}\mu(\mathrm{d}x)\\ &= \exp\Big(-\frac{\sigma^2}{2}u^2 + i\gamma u + \int \big(e^{iux} - 1 - iux\mathbb{1}_{[-1,1]}(x)\big)\nu(\mathrm{d}x)\Big), \quad u \in \mathbb{R}.\end{aligned}$$

The key question of this article is under which conditions on the characteristic triplet $|\varphi(u)|$ decays polynomially for $|u| \to \infty$. Note that, first, $|\varphi(u)|$ does not depend on γ and, second, if $\sigma^2 > 0$ then $|\varphi(u)|$ is of the order $e^{-\sigma^2 u^2/2}$. Hence, we basically have to study the interplay between the Lévy measure ν and the behavior of $|\varphi(u)|$ as $|u| \to \infty$.

Due to the connection between the decay of a characteristic function φ and the regularity of the transition density of the corresponding Lévy processes established by Hartman and Wintner (1942), upper bounds for $|\varphi|$ have attracted a certain interest in the literature. Orey (1968); Kallenberg (1981) and Knopova and Schilling (2013) have studied necessary and

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sufficient conditions for an exponential decay of φ and thus for the existence of infinitely smooth transition densities. The less regular case of polynomially decaying characteristic functions is essentially studied for self-decomposable distributions only. Here, the existence of polynomial upper bounds is in detail analyzed, see Sato (1999, Chap. 28) and references therein.

While upper bounds for $|\varphi|$ are more interesting from the probabilistic perspective, lower bounds are highly relevant from a statistical point of view. Surprisingly, polynomially lower bounds are only known for several explicit parametric classes of IDDs, for instance, the family of Gamma distributions. Trabs (2014) has established a polynomial lower bound for a class of Lévy processes which is slightly larger than self-decomposable processes.

Inspired by the results on self-decomposable distributions, we will show that under a mild regularity assumption on ν in a neighborhood of the origin, φ decays polynomially if and only if there is no diffusion component and the Lebesgue density of $x\nu(\mathrm{d}x)$ (which we assume to exist at zero) is bounded. From the degree of the polynomial decay we conclude how many continuous derivatives the probability density admits and we will show that this number is sharp in the sense that there cannot be more derivatives, generalizing the result on self-decomposable distributions by Wolfe (1971).

Let us illustrate the statistical interest in lower bounds on $|\varphi|$ in the prototypical deconvolution model. We observe an i.i.d. sample

$$Y_i = X_i + \varepsilon_i, \quad i = 1, \ldots, n, \tag{1}$$

where the target population $X_1, \ldots, X_n$ is corrupted by independent additive noise ε_i. In many applications the error ε_i can be understood as an aggregation of many small independent influences. Therefore, IDDs build a natural class of error distributions since they can be characterized as the class of limit distributions of the sum of independent, uniformly asymptotically negligible random variables. Indeed, popular examples like the normal or the Laplace distribution are IDD. As shown by Fan (1991) nonparametric convergence rates for estimating the distribution of X_i depend on the decay of the characteristic function of ε_i. In particular, a polynomial decay corresponds to mildly ill-posed estimation problems allowing polynomial convergence rates which is much faster than the logarithmic rates in the case of an exponential decay. Due to the auto-deconvolution structure of discretely observed Lévy processes reported by Belomestny and Reiß (2006), estimating the characteristic triplet of a Lévy process depends on the decay of the characteristic function of the marginal IDD, too.

Since the observations Y_i are distributed according to the convolution of the distributions of X_i and ε_i, we have to divide in the Fourier domain the (estimated) characteristic function of Y_i by the characteristic function of ε_i to assess the distribution of X_i. This spectral approach gives raise to the map $f \mapsto \mathcal{F}^{-1}[\mathcal{F}f/\varphi]$. Slightly abusing notation, we consequently denote $\mathcal{F}^{-1}[1/\varphi]$ as the *deconvolution operator* which has a prominent role in the statistical analysis of the deconvolution model and related models. To analyze its mapping properties, the Fourier multiplier approach by Nickl and Reiß (2012) is extremely useful. Studying a Lévy process model, they have shown under certain sufficient assumptions that $1/\varphi$ is a Fourier multiplier on Besov spaces. We refer to Triebel (2010) for definitions and properties of the Besov spaces $B^s_{p,q}(\mathbb{R})$, $s \in \mathbb{R}$, $p, q \in [1, \infty]$. We say $1/\varphi$ satisfies the *Fourier multiplier property* if there exists some $\alpha > 0$ such that for all $s \in \mathbb{R}$, $1 \leqslant p, q \leqslant \infty$ the linear map

$$B^{s+\alpha}_{p,q}(\mathbb{R}) \ni f \mapsto \mathcal{F}^{-1}\Big[\frac{\mathcal{F}f}{\varphi}\Big] \in B^s_{p,q}(\mathbb{R}) \tag{2}$$

is bounded. In the deconvolution model the Fourier multiplier approach and the closely related pseudo-differential operators were exploited in a series of recent papers Söhl and Trabs (2012); Schmidt–Hieber et al. (2013); Dattner et al. (2014) as well as Trabs (2014) for an overview. Minimal conditions on the IDD which imply the Fourier multiplier property are desirable to be able to apply this approach in a wide area of models. It turns out that this mapping property is very natural in the context of IDDs: We show that a polynomial decay of the characteristic function is necessary and (under a mild regularity condition) sufficient to conclude that $1/\varphi$ is a Fourier multiplier on Besov spaces.

2. Polynomial decay of the characteristic function

Before we precisely state our results, let us introduce some notation. The space of finite signed Borel measures on the real line will be denoted by $\mathcal{M}(\mathbb{R})$. For any $\mu \in \mathcal{M}(\mathbb{R})$ there are two positive finite measures μ^+, μ^- such that $\mu(A) = \mu^+(A) - \mu^-(A)$ for any $A \in \mathscr{B}(\mathbb{R})$. Using this so-called Jordan-decomposition, the total variation norm on $\mathcal{M}(\mathbb{R})$ is defined by

$$\|\mu\|_{TV} := \mu^+(\mathbb{R}) + \mu^-(\mathbb{R}).$$

If a function $f : \mathbb{R} \to \mathbb{R}$ is locally integrable, it defines a distribution $T_f(\psi) = \int \psi f$ on the test function space $\mathcal{D}(\mathbb{R})$ of infinitely smooth functions with bounded support. If the distributional derivative of T_f is a finite signed measure $\rho \in \mathcal{M}(\mathbb{R})$, the function f is called (weakly) differentiable with derivative $Df := \rho$. The space of functions of bounded variation is then defined by

$$BV(\mathbb{R}) := \{f : \mathbb{R} \to \mathbb{R} \text{ locally integrable}, Df \in \mathcal{M}(\mathbb{R})\}$$

with bounded variation norm $\|f\|_{BV} := \|Df\|_{TV}$ for $f \in BV(\mathbb{R})$. The measure Df satisfies $Df((a, b]) = f(b+) - f(a+)$ for $-\infty < a < b < \infty$. We will write $\lesssim, \gtrsim$ for inequalities up to constants.

Let μ be an IDD with characteristic triplet (σ^2, γ, ν). Defining the symmetrized jump measure by

$$\nu_s(A) := \nu(A) + \nu(-A)$$