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Sharp $L^2 \log L$ inequalities for the Haar system and martingale transforms

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ABSTRACT

Let $(h_n)_{n\geq 0}$ be the Haar system of functions on [0, 1]. The paper contains the proof of the estimate

$$\int_0^1 \left|\sum_{k=0}^n \varepsilon_k a_k h_k\right|^2 \log\left|\sum_{k=0}^n \varepsilon_k a_k h_k\right| ds \leq \int_0^1 \left|\sum_{k=0}^n a_k h_k\right|^2 \log\left|e^2 \sum_{k=0}^n a_k h_k\right| ds,$$

for $n = 0, 1, 2, \ldots$. Here $(a_n)_{n\geq 0}$ is an arbitrary sequence with values in a given Hilbert space $\mathcal{H}$ and $(\varepsilon_n)_{n\geq 0}$ is a sequence of signs. The constant e^2 appearing on the right is shown to be the best possible. This result is generalized to the sharp inequality

$$\mathbb{E}|g_n|^2 \log |g_n| \leq \mathbb{E}|f_n|^2 \log(e^2|f_n|), \quad n = 0, 1, 2, \ldots,$$

where $(f_n)_{n\geq 0}$ is an arbitrary martingale with values in $\mathcal{H}$ and $(g_n)_{n\geq 0}$ is its transform by a predictable sequence with values in $\{-1, 1\}$. As an application, we obtain the two-sided bound for the martingale square function $S(f)$:

$$\mathbb{E}|f_n|^2 \log(e^{-2}|f_n|) \leq \mathbb{E}S_n^2(f) \log S_n(f) \leq \mathbb{E}|f_n|^2 \log(e^2|f_n|),$$
$$n = 0, 1, 2, \ldots.$$

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1. Introduction

Let $h = (h_n)_{n\geq 0}$ be the Haar system, i.e., the collection of functions given by

$$\begin{aligned}
&h_0 = [0, 1), \qquad h_1 = [0, 1/2) - [1/2, 1),\\
&h_2 = [0, 1/4) - [1/4, 1/2), \qquad h_3 = [1/2, 3/4) - [3/4, 1),\\
&h_4 = [0, 1/8) - [1/8, 1/4), \qquad h_5 = [1/4, 3/8) - [3/8, 1/2),\\
&h_6 = [1/2, 5/8) - [5/8, 3/4), \qquad h_7 = [3/4, 7/8) - [7/8, 1)
\end{aligned}$$

and so on. Here we have identified a set with its indicator function. A classical result of Schauder (1928) states that the Haar system forms a basis of $L^p = L^p(0, 1)$, $1 \leq p < \infty$ (throughout, the underlying measure will be the Lebesgue measure). Using an inequality of Paley (1932), Marcinkiewicz (1937) proved that the Haar system is an unconditional basis provided

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$1 < p < \infty$. That is, there is a universal finite constant c_p such that

$$c_p^{-1}\left|\left|\sum_{k=0}^n a_kh_k\right|\right|_{L^p(0,1)} \leq \left|\left|\sum_{k=0}^n \varepsilon_ka_kh_k\right|\right|_{L^p(0,1)} \leq c_p\left|\left|\sum_{k=0}^n a_kh_k\right|\right|_{L^p(0,1)} \tag{1.1}$$

for any n and any $a_k \in \mathbb{R}$, $\varepsilon_k \in \{-1, 1\}$, $k = 0, 1, 2, \ldots, n$. This result is a starting point for numerous extensions and applications: in particular, it has led to the development of the theory of singular integrals, stochastic integrals, stimulated the studies on the geometry of Banach spaces and has been extended to other areas of mathematics. In particular, the inequality (1.1) has a natural counterpart in the martingale theory. Suppose that $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space, equipped with a nondecreasing sequence $(\mathcal{F}_n)_{n\geq 0}$ of sub-σ-algebras of $\mathcal{F}$. Let $f = (f_n)_{n\geq 0}$ be an adapted real-valued martingale and let $df = (df_n)_{n\geq 0}$ stand for its difference sequence, given by $df_0 = f_0$ and $df_n = f_n - f_{n-1}$ for $n \geq 1$. So, the differences df_n are $\mathcal{F}_n$-measurable and integrable, and the martingale property amounts to saying that for each $n \geq 1$, $\mathbb{E}(df_n|\mathcal{F}_{n-1}) = 0$. Given a deterministic sequence $\varepsilon = (\varepsilon_n)_{n\geq 0}$ of signs, we define $g = (g_n)_{n\geq 0}$, the associated transform of f, by

$$g_n = \sum_{k=0}^n \varepsilon_k df_k, \quad n = 0, 1, 2, \ldots.$$

Clearly, this is equivalent to saying that the difference sequence of g is given by $dg_n = \varepsilon_n df_n$. Note that the sequence $g = (g_n)_{n\geq 0}$ is again an adapted martingale. Actually, this is still true if we allow the following more general class of the transforming sequences. Namely, suppose that $\varepsilon = (\varepsilon_n)_{n\geq 0}$ is a sequence of random signs. We say that ε is *predictable*, if for each n, the random variable ε_n is measurable with respect to $\mathcal{F}_{(n-1)\vee 0}$.

A celebrated result of Burkholder (1966) states that for any $1 < p < \infty$ there is a finite constant c_p' such that for f, g as above, we have

$$||g_n||_p \leq c_p'||f_n||_p, \quad n = 0, 1, 2, \ldots. \tag{1.2}$$

Let c_p(1.1), c_p'(1.2) denote the optimal constants in (1.1) and (1.2), respectively. The Haar system is a martingale difference sequence with respect to its natural filtration (on the probability space being the Lebesgue unit interval) and so is $(a_kh_k)_{k\geq 0}$, for given fixed real numbers $a_0, a_1, a_2, \ldots$. Therefore, $c_p(1.1) \leq c_p'(1.2)$ for all $1 < p < \infty$. It follows from the results of Burkholder (1981) and Maurey (1974) that in fact the constants coincide: $c_p(1.1) = c_p'(1.2)$ for all $1 < p < \infty$. The question about the precise value of c_p (1.1) was answered by Burkholder in Burkholder (1984): $c_p(1.1) = p^* - 1$ (where $p^* = \max\{p, p/(p-1)\}$) for $1 < p < \infty$. Furthermore, the constant does not change if we allow the martingales and the terms a_k to take values in a separable Hilbert space $\mathcal{H}$.

One can study various sharp extensions and modifications of the estimates (1.1) and (1.2). These include the weak-type (p, p) inequalities (cf. Burkholder (1984); Suh (2005)), exponential bounds (Burkholder (1991)), logarithmic estimates (Osękowski (2008)) and many others: see the monograph by Osękowski (2012) for the detailed exposition on the subject. The purpose of this paper is to continue this line of research. Our main result is the following sharp $L^2 \log L$ bound for the Haar system and martingale transforms.

Theorem 1.1. *Let f be a martingale taking values in a Hilbert space $\mathcal{H}$ and let g be its transform by a predictable sequence of signs. Then we have the estimate*

$$\mathbb{E}|g_n|^2 \log|g_n| \leq \mathbb{E}|f_n|^2 \log(e^2|f_n|), \quad n = 0, 1, 2, \ldots. \tag{1.3}$$

This inequality is already sharp for the Haar system: for any $\kappa < e^2$ there exists a positive integer n, real numbers $a_0, a_1, \ldots, a_n$ and signs $\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_n$ such that

$$\int_0^1 \left|\sum_{k=0}^n \varepsilon_ka_kh_k\right|^2 \log\left|\sum_{k=0}^n \varepsilon_ka_kh_k\right| dx > \int_0^1 \left|\sum_{k=0}^n a_kh_k\right|^2 \log\left|\kappa\sum_{k=0}^n a_kh_k\right| dx.$$

Actually, it will be clear from the proof that the estimate (1.3) holds true for any martingales f, g satisfying the condition $|df_n| = |dg_n|$ almost surely for all $n = 0, 1, 2, \ldots$. Of course, this condition is satisfied if g is a transform of f by a predictable sequence of signs; however, generally, this new requirement is much less restrictive and, in particular, it will allow us to obtain an interesting two-sided bound for the martingale square function $S(f) = (S_n(f))_{n\geq 0}$, defined by

$$S_n(f) = \left(\sum_{k=0}^n |df_k|^2\right)^{1/2}, \quad n = 0, 1, 2, \ldots.$$

The result can be stated as follows.

Theorem 1.2. *Let f be a martingale taking values in a Hilbert space $\mathcal{H}$. Then for any $n = 0, 1, 2, \ldots$ we have*

$$\mathbb{E}|f_n|^2 \log(e^{-2}|f_n|) \leq \mathbb{E}S_n^2(f) \log S_n(f) \leq \mathbb{E}|f_n|^2 \log(e^2|f_n|). \tag{1.4}$$

The left inequality is sharp: the constant e^{-2} cannot be replaced by a larger number.

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