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Computing subsignatures of systems with exchangeable component lifetimes

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ABSTRACT

The subsignatures of a system with continuous and exchangeable component lifetimes form a class of indexes ranging from the Samaniego signature to the Barlow–Proschan importance index. These indexes can be computed through explicit linear expressions involving the values of the structure function of the system. We show how the subsignatures can be computed more efficiently from the reliability function of the system via identifications of variables, differentiations, and integrations.

1. Introduction

Consider an n-component system (C, ϕ), where C is the set $\{1, \ldots, n\}$ of its components and $\phi: \{0, 1\}^n \to \{0, 1\}$ is its structure function which expresses the state of the system in terms of the states of its components. We assume that the system is semicoherent, which means that ϕ is nondecreasing in each variable and satisfies the conditions $\phi(0, \ldots, 0) = 0$ and $\phi(1, \ldots, 1) = 1$. We also assume that the components have continuous and exchangeable lifetimes $T_1, \ldots, T_n$.

Marichal (2014) recently introduced the concept of subsignature of a system as follows. Let M be a nonempty subset of the set C of components and let $m = |M|$. The *M-signature* of the system is the m-tuple $\mathbf{s}_M = (s_M^{(1)}, \ldots, s_M^{(m)})$, where $s_M^{(k)}$ is the probability that the kth failure among the components in M causes the system to fail. That is,

$$s_M^{(k)} = \Pr(T_S = T_{k:M}), \quad k \in \{1, \ldots, m\},$$

where T_S and $T_{k:M}$ denote, respectively, the lifetime of the system and the kth smallest lifetime of the components in M, i.e., the kth order statistic obtained by rearranging the variables T_i $(i \in M)$ in ascending order of magnitude. A *subsignature* of the system is an M-signature for some $M \subseteq C$.

When $M = C$ the M-signature reduces to the *signature* $\mathbf{s} = (s_1, \ldots, s_n)$ of the system, a concept introduced by Samaniego (1985) to compare different system designs and to easily compute the reliability of any system.[1] When M is a

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[1] Actually, Samaniego (1985) proved that, when the component lifetimes are independent and identically distributed, the system reliability can always be expressed as the sum of the order statistics distributions weighted by the signature $\mathbf{s}$; this result was then established by Navarro and Rychlik (2007) in the more general case of exchangeable lifetimes.

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singleton $\{j\}$ the M-signature reduces to the 1-tuple $s^{(1)}_{\{j\}} = \Pr(T_S = T_j)$, which is the *Barlow–Proschan index* for component j, a concept introduced by Barlow and Proschan (1975) to measure the importance of the components. Thus, the subsignatures define a class of $2^n - 1$ indexes that range from the standard signature (when $M = C$) to the Barlow–Proschan index (when M is a singleton).

The M-signature of a system can be computed through any of the following explicit formulas (see Marichal, 2014)[2]:

$$s^{(k)}_M = \sum_{\substack{A\subseteq C\\ |M\cap A|=m-k+1}} \frac{m-k+1}{n\binom{n-1}{|A|-1}}\,\phi(A) - \sum_{\substack{A\subseteq C\\ |M\cap A|=m-k}} \frac{k}{n\binom{n-1}{|A|}}\,\phi(A), \tag{1}$$

$$s^{(k)}_M = \sum_{j\in M}\sum_{\substack{A\subseteq C\setminus\{j\}\\ |M\setminus A|=k}} \frac{1}{n\binom{n-1}{|A|}}\big(\phi(A\cup\{j\}) - \phi(A)\big). \tag{2}$$

Eqs. (1) and (2) show that, under the exchangeable assumption, the subsignatures do not depend on the distribution of the variables $T_1,\ldots,T_n$ but only on the structure function. When $M = C$, formula (1) reduces to Boland's formula (Boland, 2001)

$$s_k = \sum_{\substack{A\subseteq C\\ |A|=n-k+1}} \frac{1}{\binom{n}{|A|}}\,\phi(A) - \sum_{\substack{A\subseteq C\\ |A|=n-k}} \frac{1}{\binom{n}{|A|}}\,\phi(A).$$

When $M = \{j\}$, formula (2) reduces to Shapley–Shubik's formula (Shapley, 1953; Shapley and Shubik, 1954)

$$I^{(j)}_{\mathrm{BP}} = \sum_{A\subseteq C\setminus\{j\}} \frac{1}{n\binom{n-1}{|A|}}\big(\phi(A\cup\{j\}) - \phi(A)\big). \tag{3}$$

The computation of the subsignatures by means of Eqs. (1) and (2) may be cumbersome and tedious for large systems since it requires the evaluation of $\phi(A)$ for every $A \subseteq C$. To overcome this issue, in this paper we show how these indexes can be computed from simple manipulations of the reliability function of the structure ϕ such as identifications of variables and differentiations.

Recall that the *reliability function* of the structure ϕ is the multilinear function $h\colon [0,1]^n \to \mathbb{R}$ defined by

$$h(\mathbf{x}) = h(x_1,\ldots,x_n) = \sum_{A\subseteq C}\phi(A)\prod_{i\in A}x_i\prod_{i\in C\setminus A}(1-x_i). \tag{4}$$

When the component lifetimes are independent, this function expresses the reliability of the system in terms of the component reliabilities; see Barlow and Proschan (1981, Chap. 2) for a background on reliability functions and Ramamurthy (1990, Section 3.2) for a more recent reference. It is easy to see that this function can always be put in the standard multilinear form

$$h(\mathbf{x}) = \sum_{A\subseteq C} d(A)\prod_{i\in A}x_i, \tag{5}$$

where the link between the coefficients $d(A)$ and the values $\phi(A)$ is given through the conversion formulas

$$d(A) = \sum_{B\subseteq A}(-1)^{|A|-|B|}\,\phi(B) \quad\text{and}\quad \phi(A) = \sum_{B\subseteq A} d(B).$$

Example 1. The structure of a system consisting of two components connected in parallel is given by

$$\phi(x_1,x_2) = \max(x_1,x_2) = x_1 \amalg x_2 = x_1 + x_2 - x_1x_2\,,$$

where $\amalg$ is the (associative) coproduct operation defined by $x \amalg y = 1-(1-x)(1-y)$. Considering only the multilinear expression of function ϕ, one immediately obtains the corresponding reliability function $h(x_1,x_2) = x_1 + x_2 - x_1x_2$. $\square$

For any function f of n variables, we denote its diagonal section $f(x,\ldots,x)$ simply by $f(x)$. For instance, from Eqs. (4) and (5) we derive

$$h(x) = \sum_{A\subseteq C}\phi(A)\,x^{|A|}\,(1-x)^{n-|A|} = \sum_{A\subseteq C} d(A)\,x^{|A|}.$$

Owen (1972) observed that the right-hand expression in Eq. (3), which is the Barlow–Proschan index for component j, can be computed by integrating over $[0,1]$ the diagonal section of the jth partial derivative of h. That is,

$$I^{(j)}_{\mathrm{BP}} = \int_0^1 (\partial_j h)(t)\,dt. \tag{6}$$

[2] Here and throughout we identify Boolean vectors $\mathbf{x} \in \{0,1\}^n$ and subsets $A \subseteq \{1,\ldots,n\}$ by setting $x_i = 1$ if and only if $i \in A$. We thus use the same symbol to denote both a function $f\colon \{0,1\}^n \to \mathbb{R}$ and its corresponding set function $f\colon 2^{\{1,\ldots,n\}} \to \mathbb{R}$ interchangeably.