Statistics and Probability Letters 94 (2014) 153–161

Contents lists available at ScienceDirect

Statistics and Probability Letters

journal homepage: www.elsevier.com/locate/stapro

A note on processes with random stationary increments

Haimeng Zhang [a,*], Chunfeng Huang [b,c]

[a] Department of Mathematics and Statistics, University of North Carolina at Greensboro, Greensboro, NC 27412, USA
[b] Department of Statistics, Indiana University, Bloomington, IN 47408, USA
[c] Department of Geography, Indiana University, Bloomington, IN 47408, USA

ARTICLE INFO

Article history:
Received 7 November 2013
Received in revised form 14 May 2014
Accepted 14 July 2014
Available online 21 July 2014

Keywords:
Intrinsic stationarity
Variogram
Spectrum
Structure function
Inversion formula

ABSTRACT

When the correlation theory is considered for the processes with random stationary increments, Yaglom (1955) has developed the spectral representation theory. In this note, we complete this development by obtaining the inversion formula of the spectrum in terms of the structure function.

1. Introduction

When the process is assumed to be with random stationary *n*th increments, Yaglom (1955) has studied its correlation and developed the spectral representation theory. The results in Yaglom (1955) have profound impacts on various fields including the studies of fractional Brownian motion (Mandelbrot and Van Ness, 1968), intrinsic random function (Solo, 1992; Huang et al., 2009), and self-similar process (Unser and Blu, 2007; Blu and Unser, 2007). It also has a strong connection to the concept of generalized random process (Itô, 1954; Gel'fand, 1955; Gel'fand and Vilenkin, 1964; Yaglom, 1987).

The covariance function is used to characterize the second-order dependency. When the process is assumed to be stationary, its corresponding spectral density is widely used to describe the periodical components and frequencies (Brockwell and Davis, 2009). The spectrum and its covariance function are connected through the inversion of Fourier transformation. When the process is intrinsically stationary, the variogram often replaces the covariance function to model the dependency (Cressie, 1993). Furthermore, when the process is with stationary *n*th increments, the notion of structure equation (Yaglom, 1955) or generalized covariance function (Matheron, 1973) is developed. Such applications are essential in universal kriging (Cressie, 1993; Chilès and Delfiner, 2012). The spectral representation of such structure equation or generalized covariance has been obtained in Yaglom (1955) and Matheron (1973). In this note, we derive the inversion formula where the spectrum can be represented by the structure function. We are not aware of such an explicit formula in literature. This enhances the understanding of the process, where our theorem offers a way to estimate the spectral function from an easily estimated structure function. In addition, by directly applying our formula, one can derive the spectral density function of commonly used power variogram.

* Corresponding author. Tel.: +1 336 334 5836; fax: +1 336 334 5949.
E-mail addresses: haimengzhanguncg@gmail.com (H. Zhang), huang48@indiana.edu (C. Huang).

http://dx.doi.org/10.1016/j.spl.2014.07.016

2. Main results

In Yaglom (1955), a random process $\{X(t), t \in \mathbb{R}\}$ is considered, where its nth difference with step $\tau > 0$ is defined as

$$\Delta_\tau^{(n)} X(t) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} X(t - k\tau).$$

Such a random nth increment is called stationary if its second moment

$$\mathrm{E}\{\Delta_{\tau_1}^{(n)} X(t+s) \overline{\Delta_{\tau_2}^{(n)} X(s)}\} \equiv D^{(n)}(t; \tau_1, \tau_2)$$

exists and does not depend on s. Yaglom (1955) has termed this as the structure function of such increments, and has shown that it has the following spectral representation

$$D^{(n)}(t; \tau_1, \tau_2) = \int_{-\infty}^{\infty} e^{it\lambda} (1 - e^{-i\tau_1\lambda})^n (1 - e^{i\tau_2\lambda})^n \frac{(1+\lambda^2)^n}{\lambda^{2n}} dF(\lambda), \tag{1}$$

where the spectral function $F(\lambda)$ is a non-decreasing bounded measure. When $\tau_1 = \tau_2$, the above structure function becomes

$$D_\tau^{(n)}(t) \equiv D^{(n)}(t; \tau, \tau) = \int_{-\infty}^{\infty} e^{it\lambda} 2^n (1 - \cos(\tau\lambda))^n \frac{(1+\lambda^2)^n}{\lambda^{2n}} dF(\lambda). \tag{2}$$

In our next theorem, we obtain the inversion theorem where we express the spectral function in terms of the structure function.

Theorem 1. *Let $\lambda_1 < \lambda_2$ be the continuity points of $F(\lambda)$. We have*

$$F(\lambda_2) - F(\lambda_1) = \lim_{T\to\infty} \frac{1}{2\pi} \int_{-T}^{T} D_\tau^{(n)}(t) dt \left(\int_{\lambda_1}^{\lambda_2} e^{-i\lambda t} q(\lambda; n, \tau) d\lambda \right), \tag{3}$$

where,

$$q(\lambda) \equiv q(\lambda; n, \tau) = \frac{\lambda^{2n}}{2^n (1 - \cos(\tau\lambda))^n (1+\lambda^2)^n}.$$

Here $\tau > 0$ is selected so that the function $(1 - \cos(\lambda\tau))/\lambda^2$ has no zeros for $\lambda \in [\lambda_1, \lambda_2]$.

Remark 1. When $n = 0$, $\Delta_\tau^{(n)} X(t) = X(t)$, that is, the process is stationary itself. The structure function becomes a conventional covariance function. The inversion formula in Theorem 1 is just a regular Fourier inversion (Hannan, 1970; Doob, 1953).

The proof of Theorem 1 follows along the lines in obtaining the inversion formula for Fourier transform in literature, for example, see Hannan (1970, Chapter 2), and Doob (1953, Chapter XI). Here we denote $\phi(\lambda) = 1_{[\lambda_1,\lambda_2]}(\lambda)$ as the indicator function and let $h(\lambda) = \phi(\lambda) q(\lambda)$. Similar to Yaglom (1955, page 96), $\tau > 0$ is selected such that for $\lambda \in [\lambda_1, \lambda_2]$, $(1 - \cos(\tau\lambda))/\lambda^2$ has no zeros, and so is bounded away from zero. Therefore, there exists $C_h > 0$ such that $0 \le h(\lambda) \le C_h < \infty$, $\lambda \in \mathbb{R}$. Moreover, $h(\lambda)$ is continuous for $\lambda \in [\lambda_1, \lambda_2]$, and is absolutely integrable. Hence, its Fourier transform exists, and is given by $\hat{h}(t) = \int_{-\infty}^{\infty} e^{-i\nu t} h(\nu) d\nu$, $t \in \mathbb{R}$. The following properties for $\hat{h}(t)$ are easily obtained: $\hat{h}(-t) = \overline{\hat{h}(t)}$, $\hat{h}(t)$ is uniformly continuous and bounded for $t \in \mathbb{R}$.

Lemma 1. *For the Fourier transform function $\hat{h}(t)$ of $h(\nu)$, we have*

$$\lim_{T\to\infty} \frac{1}{2\pi} \int_{-T}^{T} \left(1 - \frac{|t|}{T}\right) e^{i\nu t} \hat{h}(t) dt = h(\nu). \tag{4}$$

Moreover, the above convergence is bounded and uniformly over all $\nu \in \mathbb{R}$.

Proof. First, since $h(\lambda)$ is absolutely integrable, from the Fubini's theorem, one has

$$\frac{1}{2\pi} \int_{-T}^{T} \left(1 - \frac{|t|}{T}\right) e^{i\nu t} \hat{h}(t) dt = \int_{-\infty}^{\infty} h(\lambda) d\lambda \left(\frac{1}{2\pi} \int_{-T}^{T} e^{i(\nu-\lambda)t} \left(1 - \frac{|t|}{T}\right) dt \right).$$

Now the above inner integral can be simplified as follows, when $\lambda \ne \nu$,

$$\int_{-T}^{T} e^{i(\nu-\lambda)t} \left(1 - \frac{|t|}{T}\right) dt = 2 \int_0^T \cos((\lambda-\nu)t) \left(1 - \frac{t}{T}\right) dt$$

$$= \frac{2}{T} \frac{1 - \cos((\lambda-\nu)T)}{(\lambda-\nu)^2} = \frac{1}{T} \left[\frac{\sin^2\left(\frac{(\lambda-\nu)T}{2}\right)}{\left(\frac{\lambda-\nu}{2}\right)^2} \right] \to 0, \quad \text{as } T \to \infty.$$