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Uniform asymptotics for the tail probability of weighted sums with heavy tails

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ABSTRACT

This paper studies the tail probability of weighted sums of the form $\sum_{i=1}^{n} c_i X_i$, where random variables X_i's are either independent or pairwise quasi-asymptotically independent with heavy tails. Using the idea of uniform long-tailedness, the uniform asymptotic equivalence of the tail probabilities of $\sum_{i=1}^{n} c_i X_i$, $\max_{1\le k\le n} \sum_{i=1}^{k} c_i X_i$ and $\sum_{i=1}^{n} c_i X_i^+$ is established, where X_i's are independent and follow the long-tailed distribution, and c_i's take value in a broad interval. Some further uniform asymptotic results for the weighted sums of X_i's with dominated varying tails are obtained. An application to the ruin probability in a discrete-time insurance risk model is presented.

1. Introduction

In this paper, all asymptotic and limit relations are taken as $x \to \infty$ unless otherwise stated. For independently and identically distributed (i.i.d.) subexponential random variables $X_i,\ i \ge 1$, it is well-known that, for any $n \ge 2$,

$$P\left(\sum_{i=1}^{n} X_i > x\right) \sim P\left(\max_{1\le k\le n} \sum_{i=1}^{k} X_i > x\right) \sim P\left(\sum_{i=1}^{n} X_i^+ > x\right) \sim \sum_{i=1}^{n} P(X_i > x), \tag{1}$$

where $x^+ = \max\{x, 0\}$. There are quite a few ways to generalize these asymptotic relations. One way is to consider some broader classes of heavy-tailed distributions, see, e.g., Ng et al. (2002). Another way is to study the randomly stopped sums, see, e.g., Denisov et al. (2010). Allowing some dependence of X_i's, similar results can be obtained for different classes of heavy-tailed distributions, see Wang and Tang (2004), Geluk and Ng (2006), Tang (2008), Geluk and Tang (2009), and references therein.

A more general way is to work on the weighted sums of form $\sum_{i=1}^{n} c_i X_i$, where weights c_i's are real numbers. If X_i's are i.i.d. subexponential random variables, Tang and Tsitsiashvili (2003) proved that for any $0 < a \le b < \infty$, the asymptotic relation

$$P\left(\sum_{i=1}^{n} c_i X_i > x\right) \sim \sum_{i=1}^{n} P(c_i X_i > x), \tag{2}$$

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holds uniformly for $a \le c_i \le b$, $1 \le i \le n$, in the sense that

$$\lim_{x\to\infty} \sup_{a\le c_i\le b, 1\le i\le n} \left| \frac{P\left(\sum_{i=1}^{n} c_i X_i > x\right)}{\sum_{i=1}^{n} P(c_i X_i > x)} - 1 \right| = 0.$$

Recently, Liu et al. (2012) and Li (2013) established the same asymptotic relation for some dependent X_i's.

Chen et al. (2011) showed that for any fixed $0 < a \le b < \infty$ it holds that uniformly for $a \le c_i \le b$, $1 \le i \le n$,

$$P\left(\sum_{i=1}^{n} c_i X_i > x\right) \sim P\left(\max_{1\le k\le n} \sum_{i=1}^{k} c_i X_i > x\right) \sim P\left(\sum_{i=1}^{n} c_i X_i^+ > x\right), \tag{3}$$

where X_i's are independent, not necessarily identically distributed, random variables with long-tailed distributions. This result is extended by substituting b with any positive function $b(x)$ such that $h(x) \nearrow \infty$ and $b(x) = o(x)$ in this paper.

Replacing the constant weights c_i's with random weights θ_i's, the asymptotic relations (2) and (3) still hold if the weights θ_i's, independent of X_i's, are uniformly bounded away from zero and infinity. Then it is very natural to consider the randomly weighted sum of form $\sum_{i=1}^{n} \theta_i X_i$. Wang and Tang (2006) obtained $P\left(\sum_{i=1}^{n} \theta_i X_i > x\right) \sim P\left(\max_{1\le k\le n} \sum_{i=1}^{k} \theta_i X_i > x\right) \sim P\left(\sum_{i=1}^{n} \theta_i X_i^+ > x\right)$ for the case that the random weights are not necessarily bounded and X_i's are independently random variables with common distribution belonging to a smaller class than the class of subexponential distributions. Furthermore, Zhang et al. (2009), Chen and Yuen (2009) established the same results for dependent X_i's, where the dependence structures of X_i's are essentially same for proof of their results.

The rest of this paper is organized as follows. Section 2 reviews some important classes of heavy-tailed distributions. Section 3 introduces the family of uniformly long-tailed distributions. Section 4 states the main results along with some corollaries. Section 5 gives an application of the main results to the ruin probability in a discrete-time insurance risk model. The proof of the main results and some lemmas are presented in Section 6.

2. Classes of heavy-tailed distributions

A random variable X or its distribution F is said to be heavy-tailed to the right or have a heavy (right) tail if the corresponding moment generate function does not exist on the positive real line, i.e., $Ee^{tX} = \int_{-\infty}^{\infty} e^{tx} dF(x) = \infty$ for any $t > 0$. The most important class of heavy-tailed distributions is the class of subexponential distributions, denoted by $\mathcal{S}$. Write the tail distribution by $\overline{F}(x) = 1 - F(x)$ for any distribution F. Let F^{*n} denote the n-fold convolution of F. A distribution F concentrated on $[0, \infty)$ is subexponential if

$$\overline{F^{*n}}(x) \sim n\overline{F}(x)$$

for some or, equivalently, for all $n \ge 2$. More generally, a distribution F on $(-\infty, \infty)$ belongs to the subexponential class if $F^+(x) = F(x)I_{\{x\ge 0\}}$ does.

Closely related to the subexponential class $\mathcal{S}$, the class $\mathcal{D}$ of dominated varying distributions consists of distributions satisfying

$$\limsup_{x\to\infty} \frac{\overline{F}(yx)}{\overline{F}(x)} < \infty$$

for some or, equivalently, for all $0 < y < 1$. A slightly smaller class of $\mathcal{D}$ is the class of distributions with consistently varying tail, denoted by $\mathcal{C}$. Say that a distribution F belongs to the class $\mathcal{C}$ if

$$\lim_{y\searrow 1} \liminf_{x\to\infty} \frac{\overline{F}(yx)}{\overline{F}(x)} = 1 \quad \text{or, equivalently,} \quad \lim_{y\nearrow 1} \limsup_{x\to\infty} \frac{\overline{F}(yx)}{\overline{F}(x)} = 1.$$

A distribution F belongs to the class $\mathcal{L}$ of long-tailed distributions if

$$\lim_{x\to\infty} \frac{\overline{F}(x+y)}{\overline{F}(x)} = 1$$

for some or, equivalently, for all y. A tail distribution $\overline{F}$ is called h-insensitive if $\overline{F}(x+y) \sim \overline{F}(x)$ holds uniformly for all $|y| \le h(x)$, where $h(x)$ is a positive nondecreasing function and $\lim_{x\to\infty} h(x) = \infty$. The concept of h-insensitive function is extensively used in the monograph of Foss et al. (2011). For any distribution $F \in \mathcal{L}$, it can be shown that $\overline{F}$ is h-insensitive for some positive nondecreasing function $h(x) := h_F(x)$ such that $h(x) \nearrow \infty$ and $h(x) = o(x)$, see, e.g., Lemma 6.1 in Section 6, Section 2 in Foss and Zachary (2003), Lemma 4.1 of Li et al. (2010). Consequently, $\overline{F}$ is ch-insensitive for any fixed positive real number c.

It is well-known that the proper inclusion relations

$$\mathcal{C} \subset \mathcal{D} \cap \mathcal{L} \subset \mathcal{S} \subset \mathcal{L}$$

hold, see, e.g., Embrechts et al. (1997) and Foss et al. (2011).