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Necessary and sufficient conditions for Hölder continuity of Gaussian processes

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ABSTRACT

The continuity of Gaussian processes is an extensively studied topic and it culminates in Talagrand's notion of majorizing measures that gives complicated necessary and sufficient conditions. In this note we study the Hölder continuity of Gaussian processes. It turns out that necessary and sufficient conditions can be stated in a simple form that is a variant of the celebrated Kolmogorov–Čentsov condition.

1. Introduction

In what follows X will always be a centered Gaussian process on the interval $[0, T]$. For a centered Gaussian family $\xi = (\xi_\tau)_{\tau\in\mathbb{T}}$ we denote

$$d_\xi^2(\tau, \tau') := \mathbb{E}[(\xi_\tau - \xi_{\tau'})^2],$$
$$\sigma_\xi^2(\tau) := \mathbb{E}[\xi_\tau^2].$$

To put our result in context, we briefly recall the essential results of Gaussian continuity.

One of the earliest results is a sufficient condition due to Fernique (1964): *Assume that for some positive ε, and $0 \le s \le t \le \varepsilon$, there exists a nondecreasing function Ψ on $[0, \varepsilon]$ such that $\sigma_X^2(s,t) \le \Psi^2(t-s)$ and*

$$\int_0^\varepsilon \frac{\Psi(u)}{u\sqrt{\log u}}\,\mathrm{d}u < \infty. \tag{1}$$

Then X is continuous. The finiteness of Fernique integral (1) is not necessary for the continuity. Indeed, cf. (Marcus and Shepp, 1970, Sect. 5) for a counter-example.

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Dudley (1967, 1973) found a sufficient condition for the continuity by using *metric entropy*. Let $N(\varepsilon) := N([0,T], d_X, \varepsilon)$ denote the minimum number of closed balls of radius ε in the (pseudo) metric d_X needed to cover $[0,T]$. *If*

$$\int_0^\infty \sqrt{\log N(\varepsilon)}\,\mathrm{d}\varepsilon < \infty, \tag{2}$$

then X is continuous. Like in the case of Fernique's condition, the finiteness of the Dudley integral (2) is not necessary for continuity, cf. (Marcus and Rosen, 2006, Ch 6.). However, for stationary processes (2) is necessary and sufficient.

Finally, necessary and sufficient conditions were obtained by Talagrand (1987). Denote $B_{d_X}(t,\varepsilon)$ a ball with radius ε at center t in the metric d_X. A probability measure μ on $([0,T], d_X)$ is called a *majorizing measure* if

$$\sup_{t\in[0,T]} \int_0^\infty \sqrt{\log \frac{1}{\mu\left(B_{d_X}(t,\varepsilon)\right)}}\,\mathrm{d}\varepsilon < \infty. \tag{3}$$

The Gaussian process X is continuous if and only if there exists a majorizing measure μ on $([0,T], d_X)$ such that

$$\lim_{\delta\to 0} \sup_{t\in[0,T]} \int_0^\delta \sqrt{\log \frac{1}{\mu(B_{d_X}(t,\varepsilon))}}\,\mathrm{d}\varepsilon = 0.$$

2. Main theorem

Talagrand's necessary and sufficient condition (3) for the continuity of a Gaussian process is rather complicated. In contrast, the general Kolmogorov–Čentsov condition for continuity is very simple. It turns out that for Gaussian processes the Kolmogorov–Čentsov condition is very close to being necessary for Hölder continuity:

Theorem 1. *The Gaussian process X is Hölder continuous of any order $a < H$ i.e.*

$$|X_t - X_s| \le C_\varepsilon |t-s|^{H-\varepsilon}, \quad \textit{for all } \varepsilon > 0 \tag{4}$$

if and only if there exist constants c_ε such that

$$d_X(t,s) \le c_\varepsilon |t-s|^{H-\varepsilon}, \quad \textit{for all } \varepsilon > 0. \tag{5}$$

Moreover, the random variables C_ε in (4) satisfy

$$\mathbb{E}\left[\exp\left(aC_\varepsilon^\kappa\right)\right] < \infty \tag{6}$$

for any constants $a \in \mathbb{R}$ and $\kappa < 2$; and also for $\kappa = 2$ for small enough positive a. In particular, the moments of all orders of C_ε are finite.

The differences between the classical Kolmogorov–Čentsov continuity criterion and Theorem 1 are: (i) Theorem 1 deals only with Gaussian processes, (ii) there is an ε-gap to the classical Kolmogorov–Čentsov condition and (iii) as a bonus we obtain that the Hölder constants C_ε must have light tails by the estimate (6). Note that the ε-gap cannot be closed. Indeed, let

$$X_t = f(t)B_t,$$

where B is the fractional Brownian motion with Hurst index H and $f(t) = (\log\log 1/t)^{-1/2}$. Then, by the law of the iterated logarithm due to Arcones (1995), X is Hölder continuous of any order $a < H$, but (5) does not hold without an $\varepsilon > 0$.

The proof of the first part Theorem 1 is based on the classical Kolmogorov–Čentsov continuity criterion and the following elementary lemma:

Lemma 1. *Let $\xi = (\xi_\tau)_{\tau\in\mathbb{T}}$ be a centered Gaussian family. If $\sup_{\tau\in\mathbb{T}} |\xi_\tau| < \infty$ then $\sup_{\tau\in\mathbb{T}} \mathbb{E}[\xi_\tau^2] < \infty$.*

Proof. Since $\sup_{\tau\in\mathbb{T}} |\xi_\tau| < \infty$, $\mathbb{P}[\sup_{\tau\in\mathbb{T}} |\xi_\tau| < x] > 0$ for a large enough $x \in \mathbb{R}$. Now, for all $\tau \in \mathbb{T}$, we have that

$$\begin{aligned}
\mathbb{P}\left[\sup_{\tau\in\mathbb{T}} |\xi_\tau| < x\right] &\le \mathbb{P}\left[|\xi_\tau| < x\right] \\
&= \mathbb{P}\left[\left|\frac{\xi_\tau}{\sigma_\xi(\tau)}\right| < \frac{x}{\sigma_\xi(\tau)}\right] \\
&= \frac{2}{\sqrt{2\pi}} \int_0^{x/\sigma_\xi(\tau)} e^{-\frac{1}{2}z^2}\,\mathrm{d}z \\
&\le \frac{2}{\sqrt{2\pi}} \frac{x}{\sigma_\xi(\tau)}.
\end{aligned}$$