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Diagnostic check for heavy tail in linear time series

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ABSTRACT

Justification of heavy tail is an important open problem. A systematic approach is proposed to verify heavy tail in linear time series. It consists of three parts, each of which is guided by statistical tests. The analysis is supplemented by an application to ozone concentration. The methodology has the advantage that the threshold selection is data-driven. Simulations show that test results are accurate even under model misspecification. The power is good under two heavy-tailed alternatives. The test is invariant when the time series clusters at extreme level in the study of the max-autoregressive process. It also gives a preliminary measure of tail heaviness if the underlying process is heavy-tailed.

1. Introduction

The study of extreme value statistics offers a wide variety of problems in many fields. Meteorological parameters for which the tail of the distribution is of special interest include ozone concentration. Hydrological data have alarming aspects when flood discharge and rainfall intensity overshoot a particular ecological level. These environmental problems concern global warming and ultimate climate change which have tremendous impact on the society. In finance, a large insurance claim may jeopardize the solvency of the portfolio and the financial position of the company. Risk management is intended to guard against risks from a tumble in speculative asset prices. The conclusion of an extreme value analysis is needed in terms of a tail probability and return level.

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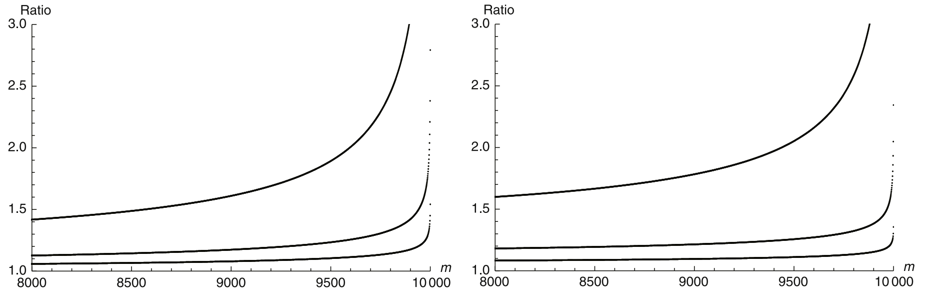

Fig. 1. Ratio of m-observation return level in $X_t = 0.5X_{t-1} + a_t - 0.2a_{t-1}$ (left) and $X_t = -0.5X_{t-1} + a_t + 0.3a_{t-1}$ (right). Degrees of freedom α in the Student's t-distribution include 2, 5 and 10 (from top to bottom).

Let $Y_1, \ldots, Y_n$ be an independent and identically distributed (i.i.d.) sequence with common distribution function F. The extreme value theory states that the maximum of Y_i when normalized by constants converges to the extreme value distribution [2,5,12] parameterized by the extreme value index γ. This is well-known as the block-maxima approach. Alternatively, the peaks-over-threshold approach is paralleled by the mathematical development of procedures based on the block-maxima approach. The distribution function of excess over a high threshold, conditional on the situation that the threshold is exceeded, is approximately the generalized Pareto (GP) distribution [24] parameterized by exactly the same extreme value index as the block-maxima approach. The case $\gamma > 0$ refers to a polynomially decreasing tail function and corresponds to a heavy-tailed parent distribution F. The case $\gamma = 0$ refers to an exponentially decreasing tail and corresponds to an exponential distribution, while $\gamma < 0$ refers to a short-tailed distribution with a finite upper endpoint. The extreme value index has a dominating effect on the tail probability and return level, especially when it is positive. Hence, degree of tail heaviness is quantified by γ and heavy tail is defined by $\gamma > 0$.

Test for $\gamma > 0$ is an important problem that remains unsolved. Consider the following autoregressive moving-average process $X_t = \phi X_{t-1} + a_t - \theta a_{t-1}$. The ratio of the m-observation return level when a_t's have the standard normal distribution to the m-observation return level when a_t's have the Student's t-distribution with α degrees of freedom is computed by simulation. The m-observation return level is a level which is exceeded on average once for every m observation. The extreme value index of the Student's t-distribution is $\gamma = 1/\alpha$ asymptotically [2, page 59]. That of the normal distribution is $\gamma = 0$ asymptotically. Fig. 1 shows two plots of the ratio, corresponding to two different time series. Different time series does not cause the ratio to change as significantly as α which measures the degree of tail heaviness. When α is small, the m-observation return level can be severalfold of that under the standard normal distribution and even tenfold when m is getting large. For example, the ratios are both over 30 when $\alpha = 2$ and $m = 10\,000$. Given that the return level is very sensitive to the degree of tail heaviness, the influence of extreme value index has serious consequences. The government may have loose control on factory productions so that environmental protection will suffer. Researchers and engineers may be misguided by specious heavy tail and more realistic dynamics that drive the system will be overlooked. On the other hand, high tails are usually associated with data sparsity. Likelihood influence results in an estimate of γ with the order of convergence $O\left(l^{1/2}\right)$, where l is the number of block maxima and $l/n \to 0$ as $n \to \infty$ is generally assumed in the block-maxima approach. In the peaks-over-threshold approach, l can be viewed as the number of threshold excesses. By the delta method, the estimate of return level is also of order $O\left(l^{1/2}\right)$ [5, Section 3.3.3]. Imprecise estimates are an unchangeable fact inherited in tail modeling. It is therefore an imperative task to develop indicators for heavy tail before assessing estimates of quantity of interest. In the statistics literature, relatively little has been done for the test of $\gamma > 0$, particularly in time series problems. In view of many existing methods, such a test is not at all straight-forward. The moving-maximum process [8] is defined with Fréchet innovations whose distribution function is $\exp\left(-x^{-\alpha}\right)$ for $\alpha > 0$. Extension of the classical framework of linear processes includes Pareto innovations [14,22] whose