Statistics and Probability Letters 105 (2015) 47–56

Contents lists available at ScienceDirect

Statistics and Probability Letters

journal homepage: www.elsevier.com/locate/stapro

Asymptotics for a class of dependent random variables

Li-Xin Zhang, Yang Zhang *

Department of Mathematics, Zhejiang University, Hangzhou, 310027, China

ARTICLE INFO

Article history:
Received 11 December 2014
Received in revised form 21 May 2015
Accepted 25 May 2015
Available online 6 June 2015

MSC:
60F05
60F15

Keywords:
Dependent random variables
Martingale
Central limit theorem
Strong invariance principle

ABSTRACT

We consider a class of dependent random variables where the dependence structure involves a factor driven by S_n/n. Under very mild conditions for the innovation, we obtain several asymptotic results for the partial sums including basic asymptotics and strong limit theorems. The fact that the asymptotic properties differ strikingly in a neighbour of a critical point makes the model very interesting.

1. Introduction

Consider a sequence of random variables $\{X_k, k \geq 1\}$ where X_k are dependent in the following way: besides the innovation, the averaged value achieved to that point will also have an impact on the next variable. Precisely speaking, $X_1 = u_1$ and for $k \geq 1$,

$$X_{k+1} = \theta \frac{S_k}{k} + u_{k+1}, \tag{1}$$

where $S_n = \sum_{k=1}^{n} X_k$, $\theta \geq 0$ is the dependence parameter and $\{u_k, k \geq 1\}$ is a sequence of martingale differences with filtrations $\mathcal{F}_k$ satisfying some rather mild conditions.

Variables with dependent structure received much attention in probability and statistics because it is dependent variables that work well for most of the models in practice. There have already been various kinds of dependent variables including the strong mixing sequence, the ARMA model, linear processes, to name some of them. We recommend Dedecker et al. (2007) and references therein for an extensive survey on dependent sequences.

Our model (1) is an analogue of the famous AR(1) model where S_k/k has replaced X_k. Since the averaged value S_n/n is always important and reflects the information about the expectation, it is reasonable that in most cases, S_n/n plays an important role in the information achieved to that point, which means S_n/n may have influence on the next trail in some way. So it is very natural to consider an impact driven by S_n/n in the dependence structure. That is where our motivation arises and we will mainly focus on a linear structure about S_n/n as displayed in (1). The parameter θ controls the power of the factor S_n/n and as we shall see in the following sections, the limit theorems for S_n are quite different with different

* Corresponding author.
E-mail addresses: stazlx@zju.edu.cn (L.-X. Zhang), statzhangyang@zju.edu.cn (Y. Zhang).

http://dx.doi.org/10.1016/j.spl.2015.05.018
0167-7152/

values of θ. This model is not stationary in general and when $\theta = 0$, the model reduces to the classical case of martingale differences which has already been discussed systematically.

We should point out that our motivation of studying model (1) also comes from Drezner and Farnum (1993). They proposed a generalized binomial distribution where the conditional success probability is a linear combination of S_n/n and p and derived the distribution of S_n. Heyde (2004) studied its limit theorem by a martingale representation, which is further generalized by James et al. (2008) and Wu et al. (2012).

In the present paper, we will study the asymptotic theory of the partial sums S_n and derive various kinds of limit theorems including the strong law of large numbers, central limit theorem, the law of the iterated logarithm and the strong invariance principle. In general, we will consider the theorems mentioned above mainly under two kinds of structure for the innovation sequence $\{u_k, k \geq 1\}$: the i.i.d. case with mean zero and the martingale difference case.

Our paper is organized as follows: Section 2 is a list of the main results together with some remarks, while in Section 3, we give some auxiliary lemmas and the proofs. In Section 4, we summarize briefly. In the sequel, a.s. is abbreviated for almost surely, $a_n \sim b_n$ means $a_n/b_n \to 1$ and $\xrightarrow{d}$ denotes the weak convergence. All the limits are taken as $n \to \infty$ unless specified otherwise.

2. Main results

We list the asymptotic results for the partial sums S_n in this section. The first theorem shows that when $0 < \theta < 1$, $\{X_k, k \geq 1\}$ satisfies the strong law of large numbers as long as the innovation sequence $\{u_k, k \geq 1\}$ does.

Theorem 1. *If the innovation* $\{u_k, k \geq 1\}$ *satisfies* $n^{-1}\sum_{k=1}^{n} u_k \to 0$ *a.s., then almost surely,*

$$\lim_{n\to\infty} \frac{S_n}{n} = 0 \quad \text{if } 0 < \theta < 1, \tag{2}$$

$$\lim_{n\to\infty} \frac{S_n}{n\log n} = 0 \quad \text{if } \theta = 1. \tag{3}$$

Remark 1. The condition that $n^{-1}\sum_{k=1}^{n} u_k \to 0$ a.s. is very mild in practice and when $\theta = 1$, if we additionally assume that $\sup_k \mathsf{E}|u_k|^{1+\delta} < \infty$ for some $\delta > 0$, from the following Theorem 2 we can prove S_n/n converges a.s. to a properly defined random variable (non-degenerate in most cases), which means that normalization n is not enough to ensure the strong law of large numbers.

Remark 2. When $0 < \theta < 1$, if $\{u_k, k \geq 1\}$ is stationary and ergodic with a common distribution U, then the strong law of large numbers shows that X_n is asymptotically equal to U in distribution and the impact of S_n/n is $o_{a.s.}(1)$, which is negligible compared to u_n. When $\theta = 1$, S_n/n is not negligible.

By the strong law of large numbers for martingales, we easily obtain the following corollary.

Corollary 1. *If* $\{u_k, k \geq 1\}$ *is a sequence of martingale differences and there exists a* $\delta > 0$ *such that* $\sup_k \mathsf{E}|u_k|^{1+\delta} < \infty$, *then the conclusion of Theorem 1 holds.*

The strong law of large numbers has already shown that, with different values of θ, there is some difference for the asymptotic results between X_n and the traditional martingale difference sequence. What makes the difference more striking is the following central limit theorem. Before we proceed, we shall introduce a class of martingale differences proposed by Abadir et al. (2014) to allow for broader applications.

Assumption 1. $\{u_k, k \geq 1\}$ is a sequence of martingale differences with filtrations $\mathcal{F}_k$ such that $\mathsf{E}u_k^2 = \sigma^2 > 0$ for all $k \geq 1$. Moreover,

(a) $\sup_k \mathsf{E}V_k^2 < \infty$,
(b) $\max_{|j-k|\geq M} |\gamma_V(j,k)| \to 0$, as $M \to \infty$,
(c) $\sup_k \mathsf{E}u_k^2 I(|u_k| > M) \to 0$, as $M \to \infty$,

where $V_k := \mathsf{E}(u_k^2|\mathcal{F}_{k-1})$ and $\gamma_V(j,k) := \mathsf{Cov}(V_j, V_k)$.

Theorem 2. *If* $0 < \theta \leq 1/2$ *and* $\{u_k, k \geq 1\}$ *is a martingale difference sequence that is either stationary and ergodic with second moment* σ^2 *or satisfies Assumption 1, then*

$$\frac{S_n}{\sqrt{n}} \xrightarrow{d} N\left(0, \frac{\sigma^2}{1-2\theta}\right) \quad \text{if } 0 < \theta < 1/2, \tag{4}$$

$$\frac{S_n}{\sqrt{n\log n}} \xrightarrow{d} N(0,\sigma^2) \quad \text{if } \theta = 1/2. \tag{5}$$