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# New independent component analysis tools for time series

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ABSTRACT

Independent component analysis is a popular approach in search of latent variables and structures in high-dimensional data. We propose extensions of classical FOBI and JADE estimates for multivariate time series, with a special focus on time series with stochastic volatility.



## 1. Introduction

Multivariate time series exhibit dependences over time as well as between variables. Especially for financial time series it is typical that the variances and covariances are random processes as well and depend on the previous history, seen as alternating periods with low and high variation. This feature is known as stochastic volatility and various modeling and prediction approaches have been suggested for such series, see for example Chapter 7 of Tsay (2014) for an overview. The number of parameters in these models increases fast with dimension and their estimation becomes hard in high dimensions. A general strategy to deal with this curse of dimensionality is to assume that the observed time series are linear combinations of latent independent or uncorrelated time series. Once the hidden independent series are identified, the multivariate estimation problem reduces to the estimation of the parameters of univariate models. For some recent work to develop these ideas, see for example Matteson and Tsay (2011) and Hu and Tsay (2014). Independent component analysis (ICA), originally developed for iid observation vectors, has recently gained some popularity as a method to find latent time series, see e.g. Broda and Paolella (2009), Chen et al. (2007), Chen et al. (2010), García-Ferrer et al. (2011) and García-Ferrer et al. (2012).

In this paper we introduce new independent component analysis methods specially designed for multivariate time series with possible stochastic volatility. To trace the ideas, we start by discussing the classical blind source separation (BSS) methods that use fourth moments and second cross-moments such as FOBI (Fourth Order Blind Identification) and JADE (Joint Approximate Diagonalization of Eigen-matrices) for iid independent component models and AMUSE (Algorithm for Multiple Unknown Signals Extraction) and SOBI (Second Order Blind Identification) for latent uncorrelated time series models. We next introduce natural extensions of the FOBI and JADE estimates applicable for time series with stochastic volatility. Finally we consider the behavior of these estimates in a simulation study. The paper ends with a brief discussion of the simulation results.

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## 2. Blind Source separation for time series

### 2.1. Notation

For convenience we use the term *time series* for the stochastic process and its realization as the context indicates which one is meant. Let $\mathbf{x} = (\mathbf{x}_t)_{t=0,\pm1,\pm2}$ denote a $p$-variate time series. For a full-rank $p \times p$ matrix $\mathbf{A}$ and a $p$-vector $\mathbf{b}$, $\mathbf{Ax} + \mathbf{b}$ is a time series with values

$$(\mathbf{Ax} + \mathbf{b})_t = \mathbf{Ax}_t + \mathbf{b}, \quad t = 0, \pm1, \pm2, \dots .$$

The mean vector and covariance matrix of a random vector $\mathbf{x}_t$ are written as $\mathrm{E}(\mathbf{x}_t)$ and $\mathrm{COV}(\mathbf{x}_t)$. $\mathrm{COV}(\mathbf{x}_t)^{-1/2}$ is a symmetric $p \times p$ matrix that satisfies $\mathrm{COV}(\mathbf{x}_t)^{-1/2}\mathrm{COV}(\mathbf{x}_t)\mathrm{COV}(\mathbf{x}_t)^{-1/2} = \mathbf{I}_p$. For a stationary time series $\mathbf{x}$, the standardized time series is written as

$$\mathbf{x}^{st} := \mathrm{COV}(\mathbf{x}_t)^{-1/2}(\mathbf{x} - \mathrm{E}(\mathbf{x}_t)).$$

A $p \times p$ matrix $\mathbf{P}$ is a permutation matrix if the linear transformation $\mathbf{x} \to \mathbf{Px}$ just permutes the elements of a $p$-vector $\mathbf{x}$. A $p \times p$ matrix $\mathbf{J}$ is a sign-change matrix if it is a diagonal matrix with diagonal elements $\pm1$. Write also $\mathcal{P}$, $\mathcal{J}$, $\mathcal{U}$ and $\mathcal{A}$ for the sets of permutation matrices, sign-change matrices, orthogonal matrices and full-rank $p \times p$ matrices, respectively. Then $\mathcal{P}, \mathcal{J} \subset \mathcal{U} \subset \mathcal{A} \subset \mathbb{R}^{p,p}$. Further, let $\mathbf{e}_i$ denote a $p$-vector where the $i$th element is 1 and all other elements zero and $\mathbf{E}^{ij} := \mathbf{e}_i\mathbf{e}_j^\top$ is a $p \times p$ matrix with 1 at entry $(i, j)$ and all other entries being zero. $\|\cdot\|$ means the Frobenius (matrix) norm, that is, $\|\mathbf{A}\| = \left(tr(\mathbf{A}^\top\mathbf{A})\right)^{1/2}$. For a square matrix $\mathbf{C}$, $\mathrm{diag}(\mathbf{C})$ be a diagonal matrix with the same diagonal elements as $\mathbf{C}$. We write also $\mathrm{off}(\mathbf{C}) = \mathbf{C} - \mathrm{diag}(\mathbf{C})$.

### 2.2. Independent component model for time series

Throughout the paper, we consider the following *independent component model* for the multivariate time series.

**Assumption 1.** The observed $p$-variate time series $\mathbf{x}$ satisfies

$$\mathbf{x} = \boldsymbol{\Omega}\mathbf{z} + \boldsymbol{\mu},$$

where $\boldsymbol{\mu} \in \mathbb{R}^p$ is a location vector, $\boldsymbol{\Omega} \in \mathcal{A}$ is a mixing matrix, and the $p$ univariate marginal time series in $\mathbf{z}$ are independent and stationary with $\mathrm{E}(\mathbf{z}_t) = \mathbf{0}$ and $\mathrm{COV}(\mathbf{z}_t) = \mathbf{I}_p$.

Notice that the assumption implies that $\mathbf{x}$ is stationary as well with $\mathrm{E}(\mathbf{x}_t) = \boldsymbol{\mu}$ and $\mathrm{COV}(\mathbf{x}_t) = \boldsymbol{\Sigma}$ where $\boldsymbol{\Sigma} := \boldsymbol{\Omega}\boldsymbol{\Omega}^\top$. The goal in blind source separation (BSS) is to find an unmixing matrix functional $\boldsymbol{\Gamma}(\mathbf{x}) \in \mathcal{A}$ and location vector functional $\boldsymbol{\mu}(\mathbf{x}) \in \mathbb{R}^p$ such that the marginal time series in $\boldsymbol{\Gamma}(\mathbf{x})(\mathbf{x} - \boldsymbol{\mu}(\mathbf{x}))$ and those in $\mathbf{z}$ are the same (up to their order and signs). Note that the argument in $\boldsymbol{\Gamma}(\mathbf{x})$ and in $\boldsymbol{\mu}(\mathbf{x})$ is the (full) probability distribution of the time series $\mathbf{x}$.

We then have the following:

**Definition 1.** A $p \times p$-matrix valued functional $\boldsymbol{\Gamma}(\mathbf{x})$ is an unmixing matrix functional if

1. $\boldsymbol{\Gamma}(\mathbf{z}) = \mathbf{JP}$ for some $\mathbf{P} \in \mathcal{P}$ and $\mathbf{J} \in \mathcal{J}$ and
2. for all $\mathbf{A} \in \mathcal{A}$ and $\mathbf{b} \in \mathbb{R}^p$, $\boldsymbol{\Gamma}(\mathbf{Ax} + \mathbf{b}) = \mathbf{JP}\boldsymbol{\Gamma}(\mathbf{x})\mathbf{A}^{-1}$ for some $\mathbf{P} \in \mathcal{P}$ and $\mathbf{J} \in \mathcal{J}$.

For an unmixing matrix functional $\boldsymbol{\Gamma}(\mathbf{x})$ it is then true, up to sign changes and permutations, that $\boldsymbol{\Gamma}(\mathbf{Ax})(\mathbf{Ax}) = \boldsymbol{\Gamma}(\mathbf{x})\mathbf{x} = \boldsymbol{\Gamma}(\mathbf{z})\mathbf{z}$. A key result for constructing unmixing matrix functionals is the following. For the proof, see for example Miettinen et al. (2015b).

**Theorem 1.** *There exists a* $\mathbf{U}_0 \in \mathcal{U}$*, such that* $\mathbf{z} = \mathbf{U}_0\mathbf{x}^{st}$.

The theorem thus states that, starting with standardized $p$-variate time series $\mathbf{x}^{st}$, one can search for a $\mathbf{U} \in \mathcal{U}$ such that $\mathbf{Ux}^{st}$ has independent component series. Then after estimating $\mathrm{E}(\mathbf{x}_t)$ and $\mathrm{COV}(\mathbf{x}_t)$, the estimation problem can be reduced to that of an orthogonal matrix $\mathbf{U}$ only. The Ghurye–Olkin–Zinger theorem (Ibragimov, 2014) implies that, if $\mathbf{z}_t$ has at most one Gaussian component then all possible solutions $\mathbf{U} \in \{\mathbf{PJU}_0 : \mathbf{P} \in \mathcal{P}, \mathbf{J} \in \mathcal{J}\}$. In the paper, we consider orthogonal matrix and unmixing matrix functionals that are based on various uses of second and fourth moments and cross-moments of the $p$-variate times series. The approaches are based on the use of

1. fourth moments of the $p$-variate random vector $\mathbf{x}_t$ (neglecting the temporal dependence),
2. autocovariances and cross-autocovariances of the $p$-variate times series $\mathbf{x}$, and
3. fourth moments and cross-moments of the $p$-variate times series $\mathbf{x}$.

In the following subsections we discuss these approaches in detail. The first two approaches are well known in the signal processing community: The first one provides classical independent component analysis tools such as FOBI and JADE and the second one AMUSE and SOBI unmixing matrix estimates. The gJADE estimate in the third approach was already proposed in the Ph.D. thesis of González Prieto (2011).