Statistics and Probability Letters 105 (2015) 136–142

Contents lists available at ScienceDirect

Statistics and Probability Letters

journal homepage: www.elsevier.com/locate/stapro

Combining inferences on the common mean of several inverse Gaussian distributions based on confidence distribution

Xuhua Liu [a,*], Na Li [b], Yuqin Hu [c,d]

[a] Department of Mathematics, China Agricultural University, Beijing 100193, China
[b] School of Management, University of the Chinese Academy of Sciences, Beijing 100049, China
[c] College of Applied Sciences, Beijing University of Technology, Beijing 100124, China
[d] School of Mathematics and Statistics, Zhejiang University of Finance & Economics, Hangzhou 310018, China

ARTICLE INFO

Article history:
Received 13 January 2015
Received in revised form 13 June 2015
Accepted 13 June 2015
Available online 19 June 2015

MSC:
62F25
62F86

Keywords:
Inverse Gaussian distribution
Confidence distribution
Coverage probability
Average interval length

ABSTRACT

In this paper, a new kind of confidence intervals for the common mean of several inverse Gaussian populations are constructed based on a combined confidence distribution. The simulation results demonstrate that the new method is very satisfactory.

1. Introduction

Since the review article of Folks and Chhikara (1978) was published, the inverse Gaussian (IG) distribution as a useful modeling tool to fit right-skewed data has received a lot of attention in many different fields, such as cardiology, pharmacokinetics, finance, life tests and remote sensing. The IG distribution was first derived by Schrödinger (1915) as the first passage time distribution of Brownian motion with a positive drift, since then many theoretical and practical researches have been demonstrated in the works of Tweedi (1957), Chhikara and Folks (1989), Seshadri (1993, 1999), Natarajan (1999), Mudholkar and Natarajan (2002), Natarajan et al. (2005), Tian (2006), Ye et al. (2010), Lin and Wu (2011), Shi and Lv (2012), etc.

The probability function of the two-parameter inverse Gaussian distribution, $IG(\mu, \lambda)$, is defined as

$$f(x|\mu,\lambda) = \left(\frac{\lambda}{2\pi x^3}\right)^{\frac{1}{2}} \exp\left\{-\frac{\lambda}{2\mu^2 x}(x-\mu)^2\right\}, \quad x > 0,\ \mu,\lambda > 0,$$

where μ is the mean parameter and λ is the scale parameter. In this paper we consider the problem of estimation of the common mean of several IG populations with unknown and unequal variances. It is a very practical problem for inferring

* Corresponding author.
E-mail address: liuxuhua@cau.edu.cn (X. Liu).

http://dx.doi.org/10.1016/j.spl.2015.06.016
0167-7152/

the common mean from several different populations with a common mean and possibly different scale parameters, for example, different instruments are utilized to measure the products produced by the same production process to estimate the average quality, different laboratories are employed to measure the amount of toxic waste in the same river, etc. Because the data in many applied areas are often positive and right-skewed, and IG distribution provides a useful tool for modeling these kinds of data, it is meaningful to research how to pool the information about a common mean from different IG populations. For this purpose, we will use confidence distribution (CD) as a main tool, take advantage of its good property for combining information, and derive a new kind of confidence intervals based on a combined CD for the common mean of several IG populations.

This article is organized as follows. Section 2 reviews the concept and some useful related conclusions about CD. In Section 3, we construct CD and combined CD for a common mean of several IG populations and give the confidence intervals of it on the basis of the combined CD. Section 4 demonstrates the simulation results. In Section 5, we illustrate the proposed new CD method with one real data set. Last section summarizes this paper.

2. A brief review of confidence distribution

The concept of confidence distribution has a long history, Cox (1958) suggested that a confidence distribution "can either be defined directly, or can be introduced in terms of the set of all confidence intervals at different levels of probability". In recent years, the confidence distribution concept has attracted a surge of renewed attention as a purely frequentist concept, and a modern definition of CD was formulated by Schweder and Hjort (2002) and Singh et al. (2005), which was pointed out as "Neymannian interpretation of Fisher's fiducial distribution".

Suppose $X_1, X_2, \ldots, X_n$ are n independent random draws from a population F and $\mathcal{X}$ is the sample space corresponding to the data set $\mathbf{X}_n = (X_1, X_2, \ldots, X_n)^T$. Let θ be a parameter of interest associated with F (F may contain other nuisance parameters), and let Θ be the parameter space. The following is the definition of confidence distribution.

Definition 1. A function $H_n(\mathbf{X}_n, \cdot) : \mathcal{X} \times \Theta \to [0, 1]$ is called a confidence distribution (CD) for a parameter θ if it satisfies the following two requirements: (i) $H_n(\cdot)$ is a continuous distribution function given $\mathbf{X}_n \in \mathcal{X}$; (ii) at the true parameter value $\theta = \theta_0$, $H_n(\theta_0) = H_n(\mathbf{X}_n, \theta_0)$, as a function of the sample $\mathbf{X}_n$, has the uniform distribution $U(0, 1)$.

The function $H_n(\cdot)$ is called an asymptotic confidence distribution (aCD) if requirement (ii) above is replaced by (ii)′, at $\theta = \theta_0$, $H_n(\theta_0) = H_n(\mathbf{X}_n, \theta_0) \xrightarrow{L} U(0, 1)$ as $n \to +\infty$, and the continuity requirement on $H_n(\cdot)$ is dropped.

Singh et al. (2007) proposed several methods to construct CD for a parameter of interest, such as using fiducial distributions, significant (p-value) functions, Bootstrap distributions and likelihood functions.

The following example illustrates how to construct a CD for the variance of a normal population.

Example 1. Suppose that we have a sample $X_i \sim N(\mu, \sigma^2)$, $i = 1, 2, \ldots, n$. Let $\overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$ and $S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \overline{X})^2$. Due to the well-known result, i.e., $\frac{(n-1)S^2}{\sigma^2} \sim \chi^2_{n-1}$, it is easy to show that $H_n(\sigma^2) = F_{\chi^2_{n-1}}\left(\frac{(n-1)S^2}{\sigma^2}\right)$ satisfies the requirements in Definition 1 and thus is a CD for σ^2, where $F_{\chi^2_{n-1}}$ denotes the cumulative distribution function of the χ^2 distribution with $n-1$ degrees of freedom.

In order to pool the information from different sources, Singh et al. (2005) developed a general recipe for combining CD functions using a coordinate-wise monotonic function from k-dimensional cube $[0, 1]^k$ to the real line $\mathbb{R} = (-\infty, +\infty)$. Assume $H_i(\theta) = H_i(\mathbf{X}_i, \theta)$, $i = 1, \ldots, k$, are CD functions for the same parameter θ from k different samples $\mathbf{X}_i$, and the sample size of $\mathbf{X}_i$ is n_i. Let $g_c(u_1, \ldots, u_k)$ be a given continuous function on $[0, 1]^k \to \mathbb{R}$ which is monotonic. Singh et al. (2005) suggested to combine the k CD functions as

$$H_c(\theta) = G_c\{g_c(H_1(\theta), \ldots, H_k(\theta))\},$$

here, the function G_c is totally determined by the monotonic g_c function: $G_c(t) = P(g_c(U_1, \ldots, U_k) \le t)$, where $U_1, \ldots, U_k$ are independent $U[0, 1]$ random variables. When the underlying true parameter values of the k individual CD functions $H_i(\theta)$s are the same, it is easy to verify that $H_c(\theta)$ is also a CD function for θ, and this function $H_c(\theta)$ contains information from all k samples, and it is referred to as a *combined CD function*. In general, we can take

$$g_c(u_1, \ldots, u_k) = F_0^{-1}(u_1) + \cdots + F_0^{-1}(u_k),$$

where $F_0(\cdot)$ is a given cumulative distribution function.

Example 2. Suppose that we have a sample $X_{ij} \sim N(\mu_i, \sigma^2)$, $i = 1, 2, \ldots, k$, $j = 1, 2, \ldots, n_i$. Let $\overline{X}_i = \frac{1}{n_i}\sum_{j=1}^{n_i} X_{ij}$ and $S_i^2 = \frac{1}{n_i-1}\sum_{j=1}^{n_i}(X_{ij} - \overline{X}_i)^2$. According to Example 1, we can obtain k CDs for the common σ^2, i.e., $H_{n_i}(\sigma^2) = F_{\chi^2_{n_i-1}}\left(\frac{(n_i-1)S_i^2}{\sigma^2}\right)$, $i = 1, 2, \ldots, k$. For pooling these CDs for the common parameter together, we use $g_c(u_1, \ldots, u_k) = \Phi^{-1}(u_1) + \cdots + \Phi^{-1}(u_k)$, where $\Phi(\cdot)$ is the cumulative distribution function of the standard normal. In this case, the combined CD for σ^2 can be expressed as $H_c(\sigma^2) = \Phi\left(\frac{1}{\sqrt{k}}[\Phi^{-1}(H_{n_1}(\sigma^2)) + \cdots + \Phi^{-1}(H_{n_k}(\sigma^2))]\right)$.