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Tests of fit for Inverse Gaussian distributions

José A. Villaseñor, Elizabeth González-Estrada *

Colegio de Postgraduados, Km. 36.5 Carr. México-Texcoco, Montecillo, 56230 México, Mexico

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ABSTRACT

A new property of the Inverse Gaussian distribution leads to a variance ratio test of fit for this model. Based on a transformation to Gamma variables, two additional tests are discussed. It turns out that the asymptotic null distributions of the tests are independent of parameters.

1. Introduction

A random variable X has an Inverse Gaussian distribution with parameters $\mu > 0$ and $\lambda > 0$, denoted by $X \sim IG(\mu, \lambda)$, if the cumulative distribution function of X is given by

$$F(x; \mu, \lambda) = \Phi\left[\sqrt{\frac{\lambda}{x}}\left(\frac{x}{\mu} - 1\right)\right] + e^{2\lambda/\mu}\Phi\left[-\sqrt{\frac{\lambda}{x}}\left(\frac{x}{\mu} + 1\right)\right], \quad x > 0,$$

where $\Phi(.)$ denotes the standard normal cdf.

It is well-known that if $X \sim IG(\mu, \lambda)$, see Johnson et al. (1994), then

(i) $E\{X\} = \mu$
(ii) $Var(X) = \mu^3/\lambda$
(iii) $\lambda \frac{(X-\mu)^2}{\mu^2 X} \sim \chi_1^2$, where χ_1^2 denotes the chi-square distribution with one degree of freedom.

Therefore, if we let $\beta = 2\frac{\mu^2}{\lambda}$, then

$$Z^{(\mu)} = \frac{(X-\mu)^2}{X} \sim G\left(\frac{1}{2}, \beta\right), \tag{1}$$

where $G\left(\frac{1}{2}, \beta\right)$ denotes the Gamma distribution with shape parameter 1/2 and scale parameter β.

The family of Inverse Gaussian distributions has applications in finance, lifetime testing, etc. In this manuscript we consider the problem of testing the IG assumption due to the relevance of this model in statistical applications. For this problem there exist several modified versions of the classical empirical distribution function (EDF) tests and some tests

* Corresponding author.
E-mail address: egonzalez@colpos.mx (E. González-Estrada).

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based on the empirical Laplace transform of the observations. Koutrouvelis and Karagrigoriou (2012) provide a complete review on existing tests for this model.

Many tests for the IG distribution rely on the use of tables containing critical values for different values of the shape parameter $\phi = \lambda/\mu$ since the null distributions of their test statistics depend on the unknown value of ϕ. Other tests rely on the use of parametric bootstrap for approximating null distributions, from which approximated p-values and/or critical values are obtained. Here we propose a variance ratio test and two additional tests of fit for the IG distribution based on property (1), which are asymptotically independent of parameter ϕ under the null hypothesis. Vexler et al. (2011) proposed a test based on an empirical likelihood ratio, which can be considered of the same type of the tests proposed here.

This manuscript is organized as follows. In Section 2 we present the proposed tests and the asymptotic null distribution of the variance ratio test is obtained. Section 3 contains the results of a simulation study conducted to compare the power functions of these tests under three classical alternative distributions for this model. Finally, some conclusions are included.

2. New tests for IG distributions

Let $X_1, \ldots, X_n$ be a random sample of size n from a cumulative distribution function F_X. Next we present three tests of fit for the composite null hypothesis:

$$H_0 : X \sim IG(\mu, \lambda), \tag{2}$$

where μ and λ are unknown.

2.1. A variance ratio test

Let $X_1, \ldots, X_n$ be a random sample from the $IG(\mu, \lambda)$ distribution, the maximum likelihood (ML) estimators of μ and λ are $\hat{\mu} = \bar{X}_n$ and $\hat{\lambda} = 1/V_n$, where $\bar{X}_n = \sum_{i=1}^n X_i/n$ is the sample mean and $V_n = \frac{1}{n}\sum_{i=1}^n (1/X_i - 1/\bar{X}_n)$. Let S_n^2 be the sample variance and notice that the ML estimator of $\sigma_X^2 = Var(X)$ is $\hat{\sigma}_X^2 = \bar{X}_n^3 V_n$.

Notice that property (1) implies $cov(X, Z^{(\mu)}) = 0$. Hence, if we estimate μ by $\bar{X}_n$ and define $Z = (X - \bar{X}_n)^2/X$, we might propose a test based on the moments estimator of $cov(X, Z)$, denoted by $\tilde{cov}(X, Z)$, such that it rejects H_0 if $\tilde{cov}(X, Z)$ is faraway from 0. However, since $\tilde{cov}(X, Z) = S_n^2 - \hat{\sigma}_X^2$, this leads us to a variance ratio test based on statistic $R_n = S_n^2/\hat{\sigma}_X^2$.

The following results provide the asymptotic null distribution of R_n.

Let $\xrightarrow{d}$ and $\xrightarrow{p}$ denote convergence in distribution and in probability.

Theorem 1. *If $X \sim IG(\mu, \lambda)$ then $\sqrt{n}(S_n^2 - \hat{\sigma}_X^2) \xrightarrow{d} N(0, 6\mu^7/\lambda^3)$, as $n \to \infty$.*

Proof. Notice that if $Z_i^{(\mu)} = (X_i - \mu)^2/X_i$, $\bar{Z}^{(\mu)} = \sum_{i=1}^n Z_i^{(\mu)}/n$ and $\xi = E\left\{Z^{(\mu)}\right\} = \mu^2/\lambda$ then

$$\frac{1}{n}\sum_{i=1}^n (X_i - \bar{X}_n)(Z_i^{(\mu)} - \bar{Z}^{(\mu)}) = S_n^2 - \bar{X}_n^3 V_n + (\bar{X}_n^2 - \mu^2)\bar{X}_n V_n,$$

since $Z_i^{(\mu)} - \bar{Z}^{(\mu)} = X_i - \bar{X}_n + \mu^2(1/X_i - \frac{1}{n}\sum_{j=1}^n 1/X_j)$. That is,

$$S_n^2 - \bar{X}_n^3 V_n = \frac{1}{n}\sum_{i=1}^n (X_i - \bar{X}_n)(Z_i^{(\mu)} - \bar{Z}^{(\mu)}) - (\bar{X}_n^2 - \mu^2)\bar{X}_n V_n. \tag{3}$$

On the other hand,

$$\begin{aligned} \frac{1}{n}\sum_{i=1}^n (X_i - \bar{X}_n)(Z_i^{(\mu)} - \bar{Z}^{(\mu)}) &= \frac{1}{n}\sum_{i=1}^n (X_i - \mu)(Z_i^{(\mu)} - \xi) - (\bar{X}_n - \mu)(\bar{Z}_n^{(\mu)} - \xi) \\ &= \frac{1}{n}\sum_{i=1}^n (X_i - \mu)(Z_i^{(\mu)} - \xi) + O_p(1/n), \end{aligned} \tag{4}$$

since, by the Central Limit Theorem (CLT), $(\bar{X}_n - \mu) = O_p(1/\sqrt{n})$ and $(\bar{Z}_n^{(\mu)} - \xi) = O_p(1/\sqrt{n})$.

Now notice that

$$(\bar{X}_n^2 - \mu^2)\bar{X}_n V_n = (\bar{X}_n - \mu)(\bar{X}_n + \mu)\bar{X}_n V_n = 2\xi(\bar{X}_n - \mu) + (\bar{X}_n - \mu)[(\bar{X}_n + \mu)\bar{X}_n V_n - 2\xi].$$

By consistency of $\bar{X}_n$ and V_n, $(\bar{X}_n + \mu)\bar{X}_n V_n \xrightarrow{p} 2\xi$. Then, by the CLT and Slutsky's Theorem,

$$(\bar{X}_n^2 - \mu^2)\bar{X}_n V_n = \frac{1}{n}\sum_{i=1}^n 2\xi(X_i - \mu) + O_p(1/\sqrt{n}). \tag{5}$$