Statistics and Probability Letters 105 (2015) 195–202

Contents lists available at ScienceDirect

Statistics and Probability Letters

journal homepage: www.elsevier.com/locate/stapro

Weighted sums of strongly mixing random variables with an application to nonparametric regression

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ARTICLE INFO

Article history:
Received 22 November 2014
Received in revised form 13 February 2015
Accepted 13 May 2015
Available online 20 June 2015

MSC:
60F15
60G35

Keywords:
Complete convergence
Strongly mixing process
Limit theorem
Weighted sum
Nonparametric regression

ABSTRACT

This note establishes complete convergence for weighted sums of strongly mixing random variables. The result obtained is sharp. If the condition is relaxed slightly, the desired complete convergence does not hold, which is illustrated by two examples. An application of the main result to nonparametric regression is also considered.

1. Introduction

Mixing random processes, whose remote past and distant future are asymptotically independent, naturally arise in a wide range of applications. Their asymptotic properties are of crucial importance. Treating weighted sums of random variables, this work focuses on strongly mixing processes. Under appropriate conditions, we derive a strong limit theorem, which provides characterizations of the underlying processes from the angle of complete convergence. Such limit theorem for weighted sums has many important applications in stochastic control and statistics. Examples include state observers, least squares estimators, nonparametric regression function estimators, and jackknife estimates, among others; see for example, Thanh et al. (2011) and the references therein.

Let $(\Omega, \mathcal{F}, P)$ be a probability space and let $\mathcal{A}$ and $\mathcal{B}$ be two sub-σ-algebras of $\mathcal{F}$. Denote by $\mathcal{L}_2(\mathcal{A})$ the space of all square integrable and $\mathcal{A}$-measurable random variables. The strong mixing coefficient, φ-mixing coefficient, and maximal coefficient of correlation are defined by $\alpha(\mathcal{A}, \mathcal{B}) = \sup\left\{|P(AB) - P(A)P(B)| : A \in \mathcal{A},\ B \in \mathcal{B}\right\}$, $\varphi(\mathcal{A}, \mathcal{B}) = \sup\left\{|P(B|A) - P(B)| : A \in \mathcal{A},\ B \in \mathcal{B},\ P(A) > 0\right\}$, and $\rho(\mathcal{A}, \mathcal{B}) = \sup_{f \in \mathcal{L}_2(\mathcal{A}),\ g \in \mathcal{L}_2(\mathcal{B})} |\mathrm{corr}(f, g)|$, respectively. Let $X := \{X_k,\ k \geq 1\}$ be a sequence of random variables. For $1 \leq J \leq L \leq \infty$, define the σ-field $\mathcal{F}_J^L = \sigma(X_k,\ J \leq k \leq L)$. Here and hereafter, for a subset S of $\mathbb{N} = \{1, 2, \ldots\}$, $\sigma(S)$ denotes the σ-field generated by $\{X_n,\ n \in S\}$. For $k \geq 1$, define $\alpha_k = \sup_{j \geq 1} \alpha(\mathcal{F}_1^j, \mathcal{F}_{j+k}^\infty)$, $\varphi_k =$

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http://dx.doi.org/10.1016/j.spl.2015.05.022
0167-7152/

$\sup_{j\geq 1}\varphi(\mathcal{F}_1^j,\mathcal{F}_{j+k}^\infty)$, $\rho_k=\sup_{j\geq 1}\rho(\mathcal{F}_1^j,\mathcal{F}_{j+k}^\infty)$, and $\rho_k^*=\sup\rho(\sigma(S),\sigma(T))$, where the sup is taken over all pairs of nonempty finite sets S,T of $\mathbb{N}$ such that $\mathrm{dist}(S,T)\geq k$. The sequence $\{X_i,\ i\geq 1\}$ is said to be strongly mixing (respectively, φ-mixing, ρ-mixing, ρ^*-mixing) if $\alpha_k\to 0$ (respectively, $\varphi_k\to 0,\ \rho_k\to 0,\ \rho_k^*\to 0$) as $k\to\infty$.

In many applications, the mixing rates are difficult to estimate; see for example, Thanh et al. (2011). Therefore it is interesting to replace the mixing rates by sufficient conditions imposed to other dependence coefficients that might be easier to verify in certain cases. The weaker mixing condition is based on the following coefficient. For $k\geq 1$, let $r_k^*=\sup|\mathrm{corr}(V;W)|$, where the supreme is taken over all nonempty finite subsets S,T of $\{1,2,\ldots\}$ such that $\mathrm{dist}(S,T)\geq k$ and all the linear combinations $V=\sum_{i\in S}a_iX_i$ and $W=\sum_{i\in T}b_iX_i$. For stationary Gaussian sequences, the coefficients ρ_k^* and r_k^* are identical (Kolmogorov and Rozanov, 1960). Throughout the paper, we use the symbols C,K to denote a generic positive constants, whose values may be different for different appearances. For $x>0$, let $\lfloor x\rfloor$ denote the greatest integer less than or equal to x and let $\log x$ denote $\log_2 x$.

In this paper, by considering weighted sums of strongly mixing sequences, we provide sufficient conditions for complete convergence. Our Theorem 2.2 extends the main result of Shao (1993) to the case of weighted sums. Two illustrative examples are provided to show that the conditions on the weights and required moments cannot be weakened. For complete convergence for other type of mixing dependence, see Peligrad and Gut (1999), Sung (2013), Wu et al. (2014) for ρ^*-mixing processes, and Shen et al. (2014) for φ-mixing processes. The scaling factor $n^{1/r}(\log n)^\beta$ was used in Sung (2013) and Wu et al. (2014), whereas we use $n^{1/r}$ instead. Also our proofs are different from theirs; see also the comments after Lemma 2.1.

The rest of the paper is arranged as follows. Section 2 presents complete convergence for weighted sums. Section 3 proceeds with an application to demonstrate the main result.

2. Complete convergence for weighted sums

The concept of complete convergence was introduced in Hsu and Robbins (1947). A sequence of random variables $\{X_n,\ n\geq 1\}$ is said to *converge completely* to 0 if

$$\sum_{n=1}^{\infty}P\big(|X_n|>\varepsilon\big)<\infty\quad\text{for all }\varepsilon>0. \tag{2.1}$$

Based on this definition, Hsu and Robbins (1947) and Erdös (1949) derived a necessary and sufficient condition for a sequence of independent identically distributed (i.i.d.) random variables $\{X_n,\ n\geq 1\}$. Their assertion reads: $EX_1=0$ and $E|X_1|^2<\infty$ if and only if

$$\sum_{n=1}^{\infty}P\Big(\Big|\sum_{i=1}^{n}X_i\Big|>\varepsilon n\Big)<\infty\quad\text{for all }\varepsilon>0. \tag{2.2}$$

Their theorem was extended by Baum and Katz (1965) as follows: Let $\eta>1/2,\ p\geq 1$, and $\{X_n,\ n\geq 1\}$ be a sequence of i.i.d. random variables. Then $EX_1=0$ and $E|X_1|^p<\infty$ if and only if

$$\sum_{n=1}^{\infty}n^{\eta p-2}P\Big(\Big|\sum_{i=1}^{n}X_i\Big|>\varepsilon n^\eta\Big)<\infty\quad\text{for all }\varepsilon>0. \tag{2.3}$$

Motivated by applications in sequential analysis and renewal theory, many researchers extended the complete convergence to weakly dependent random variables, including ρ-mixing (Shao, 1995) and strongly mixing (Berbee, 1987). Shao (1993) extended the result of Berbee (1987) from bounded random variables to unbounded random variables. In this section, we establish the complete convergence for weighted sums of strongly mixing random variables. The following lemma is a maximal type inequality for strongly mixing random variables; see Shao (1993) for a proof.

Lemma 2.1. *Let $\{X_i,\ 1\leq i\leq n\}$ be a sequence of zero-mean random variables. Then*

$$P\Big(\max_{j\leq n}\Big|\sum_{i=1}^{j}X_i\Big|\geq x\Big)\leq\frac{4}{x}\sum_{i=1}^{n}E|X_i|I(|X_i|>c)+\frac{4}{x^a}+\frac{32^3nc\alpha_k}{x} \tag{2.4}$$

for any $a\geq 1,\ x\geq 1,\ c>0$ and integer k satisfying the following conditions

$$1\leq k\leq\frac{x}{64ac\log(x\vee 2)}, \tag{2.5}$$

and

$$\Big(\sum_{i=1}^{n}E(|X_i|^sI(|X_i|\leq c))^{2/s}\Big)\Big(\sum_{i=0}^{k}\alpha_i^{1-2/s}\Big)\leq\frac{x^2}{32^3a\log(x\vee 2)}\quad\textit{for some } s\geq 2. \tag{2.6}$$

A main tool of the proof of complete convergence theorems for weighted sums is the Rosenthal-type inequality (see, e.g., Rosenthal, 1970) for maximum partial sums as pointed out by Sung (2013). Zhou and Lin (2013) used the Rosenthal-