Statistics and Probability Letters 105 (2015) 203–208

Contents lists available at ScienceDirect

Statistics and Probability Letters

journal homepage: www.elsevier.com/locate/stapro

On descents after maximal values in samples of discrete random variables

Yu. Yakubovich

Faculty of Mathematics and Mechanics, St. Petersburg State University, Universitetsky pr. 28, Peterhof, St. Petersburg 198504, Russia

ARTICLE INFO

Article history:
Received 9 February 2015
Received in revised form 14 June 2015
Accepted 17 June 2015
Available online 25 June 2015

Keywords:
Asymptotic approximation
Maximum
Descent

ABSTRACT

We show that the expected value of the descent after the first maximum in a sample of i.i.d. discrete random variables, as the sample size grows, behaves asymptotically up to vanishing terms as the expectation of the maximal value minus the expectation of a sampled random variable, provided the latter is finite. We also show that the expected value after the last maximum exhibits the same behaviour, although it is in general slightly bigger in mean.

1. Introduction

Let $(X_1, \dots, X_n)$ be a sample of positive integer-valued random variables, i.e. a list of independent and identically distributed (i.i.d.) random variables with values in $\mathbb{N} = \{1, 2, \dots\}$. Let $M_n = \max\{X_1, \dots, X_n\}$ be the maximal value of the sample. For discrete random variables it is attained one or several times. For each occurrence of the maximal value, say $X_k = M_n$, we call $X_k - X_{k+1}$ a descent after a maximum. In general, there may be several maxima in the sample, and hence a random number of descents after maxima, but in any sample one can define the descents after the first and after the last maxima to be the descents after the left-most and after the right-most occurrences of the maximal value, correspondingly. In the case when the maximum is attained only once these two descents coincide, but in general they can be different. We are interested in the expected values of these descents.

Our interest is motivated by a recent paper (Archibald et al., 2015), where exact asymptotics (up to $o(1)$) of the mean values of these descents were found, as $n \to \infty$, for geometric i.i.d. random variables X_i, with $\mathbf{P}(X_1 = j) = pq^{j-1}$ for $j \geq 1$ and $p + q = 1$. It is shown, in particular, that the mean values of the descents after the first and the last maxima are asymptotically the same and exhibit the same fluctuations of small amplitude as the expectation of maximum itself, found in Szpankowski and Rego (1990).

Actually, the above definitions of the descents after the first and the last maxima are incomplete, because it may happen that the maximal value is attained at the last position and then the descent is not determined by the sample. Archibald et al. (2015) used a (quite natural) definition that in this case the descent is equal to the maximum, that is to say $X_{n+1} \equiv 0$. We note that other definitions are also possible, and show that, under natural assumptions, redefining the descent after the maximum occurring at the last position does not affect the $n \to \infty$ asymptotics of its mean, up to $o(1)$. It allows us to use various definitions in different cases, whatever is more appropriate. Some of possible redefinitions of the descents are proposed in Sections 2 and 4.2.

Throughout this note we assume that the mean of X_1 is finite:

$$\mathbf{E}[X_1] < \infty. \tag{1}$$

E-mail address: y.yakubovich@spbu.ru.

http://dx.doi.org/10.1016/j.spl.2015.06.020
0167-7152/

It is well known, see, e.g. David and Nagaraja (2003, Ch. 3), that (1) is the necessary and sufficient condition for the finiteness of $\mathbf{E}[M_n]$. We also assume that the support of the distribution of X_1 is an infinite subset of natural numbers. The complementary case is easier and is not considered here in order to avoid many trivial complications. Nevertheless it can be checked, either directly or by some approximation procedure, that all statements made below are still true for $\mathbb{N}$-valued i.i.d. random variables with finitely supported distribution.

For samples of infinitely supported positive integer random variables satisfying (1) we show by a direct and elementary calculation that the mean values of descents after maxima, both the first and the last, behave asymptotically similar to $\mathbf{E}[M_n - X_1]$ as $n \to \infty$, up to $o(1)$ terms (Theorem 2). We also show that the descent after the last maximum is not less in mean than after the first one (Proposition 1), although the difference vanishes as $n \to \infty$.

It is worse noticing that the exact asymptotics of the mean of the maximum of n i.i.d. discrete variables is not investigated in full detail. To the best of our knowledge, just for two families of distributions the asymptotics up to $o(1)$ is known. The first one is the aforementioned case of geometric X_i's, which was investigated in (Szpankowski and Rego, 1990), see also (Eisenberg, 2008) where an alternative approach was used. The second one is negative binomial X_i's which is in fact a generalization of the first one, explored in (Grabner and Prodinger, 1997). In each case the mean value of the maximum of n i.i.d. random variables behaves in a similar way: it is a sum of a regularly growing function $f(n)$ which grows roughly as $\text{const} \cdot \log n$, a periodic part of small amplitude depending on $f(n)$, and a vanishing remainder term. It is not known whether this type of behaviour is always the case. Actually, this behaviour is even hard to formalize, unless you examine the proof and learn that the main and periodic parts come from different (correspondingly real and imaginary) poles of a complex function, in course of an asymptotic evaluation of certain complex integral. Nevertheless, the appearance of the periodic component is quite common in problems concerning big samples of discrete random variables, see e.g. Baryshnikov et al. (1995), Bogachev et al. (2008), Janson (2006) and Szpankowski (2001).

2. Different definitions for descents at the end of the sample

As it was mentioned in the introduction, it is possible to define the descent after the maximum at the last position in various ways. We shall consider several definitions and show that the $n \to \infty$ asymptotics of the mean does not change with a change of a definition, up to $o(1)$ terms, under some natural restrictions. To formalize things, let us consider two events

$$A_n = \{X_n = M_n,\ X_i < M_n \text{ for } i = 1, \dots, n-1\},$$
$$B_n = \{X_n = M_n\},$$

that is "the first maximum is attained at the last position" and "the maximum is attained at the last position". Clearly, $\mathbf{P}[A_n] \le \mathbf{P}[B_n]$. Denote the positions of the first and the last maxima by

$$\alpha = \min\{i : X_i = M_n\},$$
$$\beta = \max\{i : X_i = M_n\}$$

and define

$$\begin{aligned} F_n &= (X_\alpha - X_{\alpha+1})(1 - \mathbf{1}_{A_n}) + M_n \mathbf{1}_{A_n}, \\ L_n &= (X_\beta - X_{\beta+1})(1 - \mathbf{1}_{B_n}) + M_n \mathbf{1}_{B_n}, \end{aligned} \tag{2}$$

where $\mathbf{1}_A$ is the indicator of the event A. These are the descents after the first and the last maxima examined in Archibald et al. (2015) for the case of geometric X_i's.

For each n consider a sequence of integer-valued random variables Z_h, $h = 1, 2, \dots$, possibly dependent on the sample $(X_1, \dots, X_n)$. We suppress the dependence on n in the notation to keep formulas readable; moreover, in many examples it is the same sequence for all n. Given the maximum equals h and is attained at the last position n we think of Z_h as of a descent following this maximum. This leads to an alternative definition

$$\begin{aligned} F_n^Z &= (X_\alpha - X_{\alpha+1})(1 - \mathbf{1}_{A_n}) + Z_{M_n} \mathbf{1}_{A_n}, \\ L_n^Z &= (X_\beta - X_{\beta+1})(1 - \mathbf{1}_{B_n}) + Z_{M_n} \mathbf{1}_{B_n}. \end{aligned} \tag{3}$$

Hence the original definition (2) of Archibald et al. (2015) corresponds to the degenerate case $Z_h = h$ almost sure (a.s.). We propose some other definitions which can be useful in appropriate contexts. For instance, one can neglect the descent following the last position by putting $Z_h \equiv 0$; we denote the descents after the first and the last maxima defined this way by F_n^0 and L_n^0, correspondingly. Another possibility is to consider a cyclic descent and take $Z_h = h - X_1$. Yet another possibility leads to a particularly simple joint distribution of M_n and F_n^Z or L_n^Z, it is explained below, see Proposition 2.

We shall show that under a natural assumption that

$$0 \le Z_h \le h \quad \text{a.s.} \tag{4}$$

the exact definition of Z_h is irrelevant to the question about the $n \to \infty$ asymptotics of $\mathbf{E}[F_n^Z]$ and $\mathbf{E}[L_n^Z]$ up to vanishing terms: