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Estimation of the bispectrum for locally stationary processes

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ABSTRACT

We consider the problem of estimating the bispectrum of a locally stationary process. A nonparametric, lag-window type estimator is considered and its asymptotic properties are investigated. As a possible application, a test for linearity in the framework of locally stationary processes is discussed.

1. Introduction

Although the assumption of covariance-stationarity is extremely popular in the analysis of discrete time series (because it allows for an elegant asymptotic theory), it is usually not justified in practice. In the reality most processes change their second-order structure over time and one concept, which incorporates this feature, is that of locally stationary processes. A triangular array $\{\mathbf{X}_T, T = 1, 2, \ldots\}$ of stochastic processes $\mathbf{X}_T = \{X_{t,T}, t = 1, 2, \ldots, T\}$ is called locally stationary if it possesses a time varying $MA(\infty)$-representation

$$X_{t,T} = \sum_{l=0}^{\infty} \psi_{t,T,l} Z_{t-l}, \quad t = 1, \ldots, T, \tag{1}$$

where the random variables Z_t are assumed to be independent and identically distributed with mean 0 and variance 1. To make the class $\mathbf{X}_T$ of processes mathematically tractable, it is commonly assumed that $X_{t,T}$ can be locally approximated by a stationary process, that is that there exist functions $\psi_l : [0, 1] \to \mathbb{R}$ such that the time varying coefficients $\psi_{t,T,l}$ are close enough to some functions $\psi_l(t/T)$, i.e., that

$$\sum_{l=0}^{\infty} \sup_{t=1,\ldots,T} |\psi_{t,T,l} - \psi_l(t/T)| = O(1/T). \tag{2}$$

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The process

$$X_t(u) := \sum_{l=-\infty}^{\infty} \psi_l(u) Z_{t-l}$$

which is obtained by replacing $\psi_{t,T,l}$ by $\psi_l(t/T)$ is called the locally approximating stationary process, that is the stationary process that approximates $X_{t,T}$ locally at the rescaled time point t/T. The above class of stochastic processes was introduced by Dahlhaus (1997) and became quite popular in recent years.

Consider now a locally stationary process as defined in (1) and assume that all moments of the innovations Z_t exist and that the locally approximating functions $\psi_l : [0, 1] \to \mathbb{R}$ from (2) are twice continuously differentiable and satisfy the following conditions,

$$\sum_{l=-\infty}^{\infty} \sup_{u\in[0,1]} |\psi_l(u)|\,|l|^2 < \infty, \tag{3}$$

$$\sum_{l=-\infty}^{\infty} \sup_{u\in[0,1]} |\psi_l'(u)|\,|l| < \infty, \tag{4}$$

$$\sum_{l=-\infty}^{\infty} \sup_{u\in[0,1]} |\psi_l''(u)| < \infty. \tag{5}$$

Then a time varying spectral density of $\mathbf{X}_T$ exists, is unique and it is given by

$$f(u, \lambda) := \frac{1}{2\pi} \sum_{l=-\infty}^{\infty} \mathbb{E}(X_t(u)X_{t+l}(u)) \exp(-i\lambda l), \quad u \in [0, 1],\ \lambda \in \mathbb{R},$$

(see Dahlhaus, 1997). Notice that under the assumptions made

$$\sup_{t,\lambda} \left| (2\pi)^{-1} |\Psi_{t,T}(e^{-i\lambda})\,|^2 - f(u, \lambda) \right| = O(T^{-1}),$$

where for $z \in C$, $\Psi_{t,T}(z) = \sum_l \psi_{t,T,l} z^l$. Similarly, we define the time varying bispectrum of the locally stationary process $\mathbf{X}$ through

$$f(u, \lambda_1, \lambda_2) := \frac{1}{(2\pi)^2} \sum_{l_1,l_2=-\infty}^{\infty} \mathbb{E}(X_t(u)X_{t+l_1}(u)X_{t+l_2}(u)) \exp(-i(\lambda_1 l_1 + \lambda_2 l_2)).$$

This definition is justified by the following, easily established property

$$\sup_{t,\lambda_1,\lambda_2} \left| (2\pi)^{-2} E(Z_1^3) \Psi_{t,T}(e^{-i\lambda_1}) \Psi_{t,T}(e^{-i\lambda_2}) \Psi_{t,T}(e^{-i(\lambda_1+\lambda_2)}) - f(u, \lambda_1, \lambda_2) \right| = O(T^{-1}).$$

In this note we consider the problem of estimating the bispectrum $f(u, \lambda_1, \lambda_2)$ based on an observed time series $X_{1,T}, X_{2,T}, \ldots, X_{T,T}$ of the locally stationary process. For this, a lag-window estimator is considered and its asymptotic properties are derived. As an application, a test of linearity is discussed.

2. A lag-window estimator and its properties

In order to estimate the bispectrum $f(u, \lambda_1, \lambda_2)$, we choose a window length N such that $N = o(T)$ holds, and define an estimator of the time-varying bispectrum through

$$\hat{f}(u, \lambda_1, \lambda_2) = \frac{1}{(2\pi)^2} \sum_{k_1,k_2=-\infty}^{\infty} w(k_1/B, k_2/B) \hat{\gamma}_N(u, k_1, k_2) \exp(-i\lambda_{1,N} k_1 - i\lambda_{2,N} k_2), \tag{6}$$

where $w : \mathbb{R}^2 \to \mathbb{R}_0^+$ is a continuously differentiable function with compact support, $\lambda_N = 2\pi N^{-1} \lfloor (2\pi)^{-1} N\lambda \rfloor$ and

$$\hat{\gamma}_N(u, k_1, k_2) := \frac{1}{N} \sum_{p=0}^{N-1} X_{\lfloor uT \rfloor - N/2+1+p} X_{\lfloor uT \rfloor - N/2+1+p+k_1} X_{\lfloor uT \rfloor - N/2+1+p+k_2},$$

(we have set $X_{t,T} = 0$ for $t \notin \{1, \ldots, T\}$). We assume that $N, B \to \infty$ and state the following assumptions on the weight function w.

Assumption 2.1. $w(\cdot, \cdot)$ has the compact support $[0, 1]^2$ with $w(0, 0)$ being equal to one, is twice continuously differentiable, and the first derivatives vanish at the origin.