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Large deviations for product of sums of random variables

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In this paper, we obtain sample path and scalar large deviation principles for the product of sums of positive random variables. We study the case when the positive random variables are independent and identically distributed and bounded away from zero or the left tail decays to zero sufficiently fast. The explicit formula for the rate function of a scalar large deviation principle is given in the case when random variables are exponentially distributed.

1. Introduction

In recent years, the limit theorems for the product of sums of random variables have generated a lot of interests in the literature, e.g. Rempala and Wesolowski (2002), Gonchigdanzan and Rempala (2006), Zhang and Huang (2007), Miao and Mu (2011) and many others. Let $(X_i)_{i=1}^{\infty}$ be a sequence of independent and identically distributed (i.i.d.) positive random variables with mean μ and variance σ^2. Let $S_k = \sum_{i=1}^{k} X_i,\ k \in \mathbb{N}$. By strong law of large numbers and Cesàro summation,

$$\left(\prod_{k=1}^{n} \frac{S_k}{k}\right)^{1/n} = e^{\frac{1}{n}\sum_{k=1}^{n}\log\frac{S_k}{k}} \to e^{\log\mu} = \mu, \quad \text{a.s.} \tag{1.1}$$

as $n \to \infty$. Rempala and Wesolowski (2002) proved the central limit theorem when X_i are i.i.d.,

$$\left(\prod_{k=1}^{n} \frac{S_k}{k\mu}\right)^{\mu/\sigma\sqrt{n}} \to e^{\sqrt{2}N(0,1)}, \tag{1.2}$$

in distribution as $n \to \infty$. Later, Gonchigdanzan and Rempala (2006) considered the central limit theorem for dependent X_i and recently Zhang and Huang (2007) proved a functional central limit theorem and Strassen's invariance principle. Miao and Mu (2011) proved a moderate deviation principle when X_i are i.i.d. In this paper, we are interested to study the large deviations. Before we proceed, recall that a sequence $(P_n)_{n\in\mathbb{N}}$ of probability measures on a topological space X satisfies the

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large deviation principle with the rate function $I : X \to \mathbb{R}$ if I is non-negative, lower semicontinuous and for any measurable set A, we have

$$-\inf_{x\in A^o} I(x) \le \liminf_{n\to\infty} \frac{1}{n}\log P_n(A) \le \limsup_{n\to\infty} \frac{1}{n}\log P_n(A) \le -\inf_{x\in\overline{A}} I(x). \tag{1.3}$$

Here, A^o is the interior of A and $\overline{A}$ is its closure. A rate function is said to be good if all the level sets $\{x : I(x) \le \alpha\}$ are compact subsets of X. We refer to Dembo and Zeitouni (1998), Varadhan (1984) for general background of the theory and the applications of large deviations.

2. Main results

Assume that $(X_i)_{i=1}^\infty$ are i.i.d. positive random variables and $\mathbb{E}[e^{\theta X_1}] < \infty$ for any $\theta \in \mathbb{R}$. Then, the Mogulskii theorem says that $\frac{1}{n}S_{\lceil nt\rceil}$ satisfies a sample path large deviation on $L_\infty[0,1]$ with the good rate function

$$I(f) = \begin{cases} \int_0^1 \Lambda(f'(t))dt & \text{if } f \in \mathcal{AC}_0^+[0,1] \\ +\infty & \text{otherwise,} \end{cases} \tag{2.1}$$

where $\Lambda(x) = \sup_{\theta\in\mathbb{R}}\{\theta x - \log \mathbb{E}[e^{\theta X_1}]\}$, $\mathcal{AC}_0^+[0,1]$ is the set of increasing, absolutely continuous functions $f(\cdot)$ with $f(0) = 0$ on $[0,1]$. The statement and the proof of the Mogulskii theorem can be found in Dembo and Zeitouni (1998). Since $(\prod_{k=1}^{\lceil nt\rceil} \frac{S_k}{k})^{1/n} = e^{\frac{1}{n}\sum_{k=1}^{\lceil nt\rceil}\log(S_k/k)}$, it is natural to use the contraction principle to prove a sample path large deviation principle in our problem. Unfortunately, the contraction principle requires the continuity of a map from the space where a large deviation principle holds to the space where you want to prove a large deviation principle but $\log(\cdot)$ has a singularity at 0^+. We can simply bypass this problem by assuming that X_i is supported on $[\epsilon,\infty)$ as in Theorem 1. Now, if we want to drop this assumption, then we need to assume that $\mathbb{P}(X_1 \le \epsilon)$ decays to zero fast enough as $\epsilon \to 0$. A result is given in Theorem 2.

Theorem 1. *Assume that $(X_i)_{i=1}^\infty$ are i.i.d. positive random variables supported on $[\epsilon,\infty)$ for some constant $\epsilon > 0$, $\mathbb{E}[e^{\theta X_1}] < \infty$ for any $\theta \in \mathbb{R}$ and $\Lambda(x) := \sup_{\theta\in\mathbb{R}}\{\theta x - \log\mathbb{E}[e^{\theta X_1}]\}$. Then, $\left(\prod_{k=1}^{\lceil nt\rceil}\frac{S_k}{k}\right)^{1/n}$ satisfies a sample path large deviation principle on $L_\infty[0,1]$ with the good rate function*

$$I(g) = \int_0^1 \Lambda\left(\left(1 + t\cdot\frac{g''g-(g')^2}{g^2}\right)e^{g'/g}\right)dt, \tag{2.2}$$

if $g(t) = e^{\int_0^t \log(f(s)/s)ds}$, $0 \le t \le 1$ for some $f \in \mathcal{AC}_0^+[0,1]$ such that $f' \ge \epsilon$ and $I(g) = +\infty$ otherwise.

Proof. Since $(\frac{1}{n}S_{\lceil nt\rceil} \in \cdot)$ satisfies a sample path large deviation on $L_\infty[0,1]$ and $(\prod_{k=1}^{\lceil nt\rceil}\frac{S_k}{k})^{1/n} = e^{\frac{1}{n}\sum_{k=1}^{\lceil nt\rceil}\log(S_k/k)}$, it is natural to use the contraction principle to prove the sample path large deviation principle in our problem. In order to apply the contraction principle, one needs to check that the map from one space to the other is continuous. We claim that if $\|f_n - f\|_{L^\infty[0,1]} \to 0$ as $n\to\infty$, then, we have

$$\sup_{0\le t\le 1}\left|\int_0^t \log\frac{f_n(s)}{s}ds - \int_0^t \log\frac{f(s)}{s}ds\right| \to 0, \tag{2.3}$$

as $n\to\infty$. Let us give some explanations. Since X_i are supported on $[\epsilon,\infty)$, we have $I(f) = +\infty$ unless $f'(s) \ge \epsilon$, $0 \le s \le 1$. Therefore, in (2.3), both $\log\frac{f_n(s)}{s}$ and $\log\frac{f(s)}{s}$ are integrable at 0^+ and hence

$$\begin{aligned}&\sup_{0\le t\le 1}\left|\int_0^t \log\frac{f_n(s)}{s}ds - \int_0^t \log\frac{f(s)}{s}ds\right| \\ &\le \int_0^\delta |\log f_n(s)|ds + \int_0^\delta |\log f(s)|ds + (1-\delta)\sup_{\delta\le s\le 1}|\log f_n(s) - \log f(s)|.\end{aligned} \tag{2.4}$$

For any $\delta' > 0$, for any sufficiently large n, we have $\epsilon s \le f_n(s) \le f(s) + \delta'$. Therefore $|\log f_n(s)| \le |\log f(s)| + |\log\epsilon s|$ and also $\|\log f_n - \log f\|_{L^\infty[\delta,1]} \to 0$ as $n\to\infty$. Therefore, letting $n\to\infty$, we have

$$\limsup_{n\to\infty}\left|\int_0^t \log\frac{f_n(s)}{s}ds - \int_0^t \log\frac{f(s)}{s}ds\right| \le \int_0^\delta |\log\epsilon s|ds + 2\int_0^\delta |\log f(s)|ds \to 0, \tag{2.5}$$

as $\delta\to 0$ since $\log f(s)$ and $\log(s)$ are integrable at 0^+.