



Smooth properties for semigroups of Lévy processes and their application



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ABSTRACT

Explicit smooth properties for the semigroup of Lévy processes are derived in terms of its symbol. As an application, we obtain new sufficient conditions for the strong Feller property of stochastic differential equations driven by the additive Lévy process.

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1. Main result

Let $Z = (Z_t)_{t \geq 0}$ be a Lévy process on $\mathbb{R}^d$, which is defined on some stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$, continuous in probability, has stationary independent increments, càdlàg trajectories, and satisfies $Z_0 = 0$, $\mathbb{P}$ -a.s. It is well known that the characteristic exponent or the symbol Φ of $(Z_t)_{t \geq 0}$, defined by

$$\mathbb{E}(e^{i\langle \xi, Z_t \rangle}) = e^{-t\Phi(\xi)}, \quad \xi \in \mathbb{R}^d,$$

enjoys the following Lévy–Khintchine representation:

$$\Phi(\xi) = \frac{1}{2} \langle Q\xi, \xi \rangle + i \langle b, \xi \rangle + \int_{z \neq 0} \left(1 - e^{i\langle \xi, z \rangle} + i \langle \xi, z \rangle \mathbb{1}_{\{|z| \leq 1\}}(z) \right) \nu(dz), \quad (1.1)$$

where $Q \in \mathbb{R}^{d \times d}$ is a positive semi-definite matrix, $b \in \mathbb{R}^d$ is the drift vector and ν is the Lévy measure, that is, a σ -finite measure on $\mathbb{R}^d \setminus \{0\}$ such that $\int_{z \neq 0} (1 \wedge |z|^2) \nu(dz) < \infty$. Our standard references for the Lévy process and its symbol are the monographs (Jacob, 2001; Sato, 1999).

Let $(P_t^Z)_{t \geq 0}$ be the semigroup associated with $(Z_t)_{t \geq 0}$. In this short paper, we are concerned with smooth properties for $(P_t^Z)_{t \geq 0}$ in terms of the symbol $\Phi(\xi)$. For this aim, we need the following notation of function spaces. Denote by $B_b(\mathbb{R}^d)$

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the set of bounded Borel measurable functions on $\mathbb{R}^d$, and by $L^p(\mathbb{R}^d)$ the standard Lebesgue space with norm $\|\cdot\|_p$ for all $p \in [1, \infty]$. For any $k \in \overline{\mathbb{N}}_0 := \mathbb{N} \cup \{0, \infty\}$, let $C_b^k(\mathbb{R}^d)$ be the space of continuously differentiable functions with bounded derivatives up to the order k ; in particular, $C_b(\mathbb{R}^d) = C_b^0(\mathbb{R}^d)$ is the space of bounded continuous functions on $\mathbb{R}^d$, and $C_b^\infty(\mathbb{R}^d)$ is the space of smooth functions with bounded derivatives for all orders. For any $k \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$ and $f \in C_b^k(\mathbb{R}^d)$, $\|f\|_{C_b^k(\mathbb{R}^d)} := \sum_{|\alpha| \leq k} \|\partial^\alpha f\|_\infty$. For a real non-integer number $s > 0$, denote by $C_b^s(\mathbb{R}^d)$ the subset of $C_b^{[s]}(\mathbb{R}^d)$ consisting of functions f such that

$$\|f\|_{C_b^s(\mathbb{R}^d)} := \sum_{|\alpha| \leq [s]} \|\partial^\alpha f\|_\infty + \sum_{|\beta|=[s]} \sup_{x \neq y} \frac{|\partial^\beta f(x) - \partial^\beta f(y)|}{|x - y|^{[s]}} < \infty,$$

where $s = [s] + \{s\}$ with $[s] \in \mathbb{N}_0$ and $0 < \{s\} < 1$.

The main contribution of this paper is as follows.

Theorem 1.1. Suppose that

$$\lim_{|\xi| \rightarrow \infty} \frac{\operatorname{Re} \Phi(\xi)}{\log(1 + |\xi|)} = \infty, \quad (1.2)$$

and there exists a constant $c > 0$ such that for any $t \in (0, 1]$ and $n \in \mathbb{N}_0$,

$$\int \exp(-t \operatorname{Re} \Phi(\xi)) |\xi|^n d\xi \leq ch(t)^{-n-d}, \quad (1.3)$$

where $h(t) = \frac{1}{\varphi^{-1}(1/t)}$ and $\varphi(\rho) = \sup_{|\xi| \leq \rho} \operatorname{Re} \Phi(\xi)$. Then, for any $p \in [1, \infty]$ and $s, t > 0$, the semigroup P_t^Z maps $L^p(\mathbb{R}^d)$ into $C_b^s(\mathbb{R}^d)$. More explicitly, for all $p \in [1, \infty]$ and $s > 0$, there is a constant $C > 0$ such that for any $t > 0$,

$$\|P_t^Z f\|_{C_b^s(\mathbb{R}^d)} \leq C \|f\|_p h(t \wedge 1)^{-s-d/p}.$$

The assumption (1.2) is called Hartman–Wintner’s condition in the literature; see Hartman and Wintner (1942) for classical results or Knopova and Schilling (2013) for a recent study. We will present some comments on Theorem 1.1. First, as a direct consequence of Theorem 1.1, we can conclude that under (1.2) and (1.3), the semigroup P_t^Z maps $B_b(\mathbb{R}^d)$ into $C_b^s(\mathbb{R}^d)$ for any s and $t > 0$. In particular, the semigroup P_t^Z is strong Feller, i.e., for any $t > 0$ and $f \in B_b(\mathbb{R}^d)$, $P_t^Z f \in C_b(\mathbb{R}^d)$. Second, Theorem 1.1 extends Kusuoka and Marinelli (in press, Proposition 3.8), where the Lévy process Z is assumed to be decomposed into two independent parts, and one of them is a subordinate Brownian motion. Third, according to Theorem 1.1 and the proof of Wang (2013a, Proposition 1.3), we have the following corollary, which reduces (1.2) and (1.3) into the asymptotic behavior of $\operatorname{Re} \Phi(\xi)$ near infinity.

Corollary 1.2. Assume that $\operatorname{Re} \Phi(\xi) \asymp g(\xi)$ as $|\xi| \rightarrow \infty$, where g is a strictly increasing function, which is differentiable on (s_0, ∞) for some constant $s_0 > 0$, and which satisfies that $\lim_{s \rightarrow \infty} g(s)/\log s = \infty$ and $\lim_{s \rightarrow \infty} g^{-1}(2s)/g^{-1}(s) < \infty$. Then, for all $p \in [1, \infty]$ and $s > 0$, there is a constant $C > 0$ such that for any $t > 0$,

$$\|P_t^Z f\|_{C_b^s(\mathbb{R}^d)} \leq C \|f\|_p \left(g^{-1} \left(\frac{1}{t \wedge 1} \right) \right)^{s+d/p}.$$

To close this section, we will present the following two examples to illustrate Theorem 1.1 and Corollary 1.2.

Example 1.3. Let Z be a subordinate Brownian motion with symbol $f(|\xi|^2)$, where $f(\lambda) = \lambda^{\alpha/2}(\log(1 + \lambda))^{\beta/2}$, $\alpha \in (0, 2)$ and $\beta \in (-\alpha, 2 - \alpha)$. Then, for all $p \in [1, \infty]$ and $s > 0$, there is a constant $C > 0$ such that for any $t > 0$,

$$\|P_t^Z f\|_{C_b^s(\mathbb{R}^d)} \leq C \|f\|_p \left\{ (t \wedge 1)^{-1} [\log(1 + (t \wedge 1)^{-1})]^{-\beta/2} \right\}^{(s+d/p)/\alpha}.$$

Example 1.4. Let μ be a finite nonnegative measure on the unit sphere $\mathbb{S}$ and assume that μ is nondegenerate in the sense that its support is not contained in any proper linear subspace of $\mathbb{R}^d$. Let $\alpha \in (0, 2)$, $\beta \in (0, \infty]$ and assume that the Lévy measure ν satisfies that for some constant $r_0 > 0$ and any $A \in \mathcal{B}(\mathbb{R}^d)$,

$$\nu(A) \geq \int_0^{r_0} \int_{\mathbb{S}} \mathbb{1}_A(s\theta) s^{-1-\alpha} ds \mu(d\theta) + \int_{r_0}^\infty \int_{\mathbb{S}} \mathbb{1}_A(s\theta) s^{-1-\beta} ds \mu(d\theta).$$

Then, for all $p \in [1, \infty]$ and $s > 0$, there is a constant $C > 0$ such that for any $t > 0$,

$$\|P_t^Z f\|_{C_b^s(\mathbb{R}^d)} \leq C \|f\|_p (t \wedge 1)^{-(s+d/p)/\alpha}.$$

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