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Construction of Sudoku designs and Sudoku-based uniform designs

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ABSTRACT

In this paper, an easy and effective construction method of Sudoku designs with any order is provided based on the right shift operator. Based on the constructed Sudoku designs, a class of Sudoku-based uniform designs is constructed. Moreover, the properties of the constructed Sudoku designs and Sudoku-based uniform designs are investigated, it is shown that both the constructed Sudoku designs and Sudoku-based uniform designs are uniform designs *in terms of* discrete discrepancy.

1. Introduction

Sudoku designs originated from a popular game which is named Sudoku puzzle. It has simple rules and is very addictive. The game board is a 9×9 grid of numbers from one to nine. Several entries within the grid are provided and the remaining entries must be filled in subject to no row, column, or 3×3 subsquare containing duplicate numbers. Sudoku puzzle is just the construction of a Sudoku design with order 9, see Table 1 for example. Now, Sudoku designs have been extensively used in many areas such as agricultural experiment, genetic statistics, biostatistics, medical statistics and so on.

Recently, many papers in terms of the construction method and the properties of Sudoku designs have been published. Subramani and Ponnuswamy (2009) considered the construction and analysis of Sudoku designs with order m^2. Subramani (2012) extended the Sudoku designs to orthogonal (Graeco) Sudoku square designs in line with that of the orthogonal (Graeco) Latin square designs. A simple method of constructing Graeco Sudoku square designs of odd orders was also presented in Subramani (2012). Xu et al. (2011) considered the construction of a space-filling design based on the Sudoku design, which can achieve maximum uniformity in univariate and bivariate margins. Fontana (2011) studied the simplification of fractional factorial design generation based on Sudoku designs. Subsequently, Fontana (2013) developed a simple algorithm for uniform random sampling of Latin squares and Sudoku designs by graph analysis tools. Meng and Lu (2011) developed an algorithm to generate puzzles which guarantees a unique solution for the most important premise and ensures low enough

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Table 1
A completed Sudoku puzzle.

1	2	3	4	5	6	7	8	9
4	5	6	7	8	9	1	2	3
7	8	9	1	2	3	4	5	6
2	3	4	5	6	7	8	9	1
5	6	7	8	9	1	2	3	4
8	9	1	2	3	4	5	6	7
3	4	5	6	7	8	9	1	2
6	7	8	9	1	2	3	4	5
9	1	2	3	4	5	6	7	8

complexity. The Sudoku design also has been studied in Fontana and Rogantin (2010) by algebraic statistics methods. Bailey et al. (2008) explained several connections between Sudoku and various parts of mathematics and statistics.

As an important method of computer experiments and robust experimental designs, uniform designs (Fang and Wang, 1994; Fang et al., 2005) distribute their experimental points evenly throughout the design space. The measure of uniformity plays a key role in the construction of uniform designs. Various discrepancies in quasi-Monte Carlo methods have been used as measures of uniformity in the literature, such as the star-discrepancy, the star-L_2-discrepancy (Fang and Wang, 1994), the centered L_2-discrepancy, the wrap-around L_2-discrepancy (Hickernell, 1998a,b; Chatterjee et al., 2005, 2006), the discrete discrepancy (Qin and Fang, 2004), and the mixture discrepancy (Zhou et al., 2013).

The aim of this paper is twofold: first, for any positive integer p, a general construction method of Sudoku designs with order p will be presented, and based on the constructed Sudoku designs, a class of Sudoku-based uniform designs are constructed; second, the properties, especially the uniformity measured by discrete discrepancy of the constructed Sudoku designs and Sudoku-based uniform designs will be investigated.

The paper is organized as follows. Section 2 provides the notations and preliminaries. In Section 3, a general construction method of Sudoku designs by right shift operator is provided, and some basic properties of the constructed Sudoku designs are investigated. A class of Sudoku-based uniform designs is constructed in Section 4, it is shown that both the constructed Sudoku designs and Sudoku-based uniform designs are uniform designs in terms of discrete discrepancy. Some illustrative examples are also provided in Section 4 to lend further support to our theoretical results. We close through the Remarks section with some notes and comments.

2. Notations and preliminaries

Definition 1. Suppose $p = m \times l$, where p, m, l are all positive integers. If the design X with order $p \times p$ satisfies that each row, each column and each $m \times l$ (or $l \times m$) block contains the numbers $1 - p$, without repeating any, then the design X is called as the Sudoku design with order p.

Consider a class of n runs and s factors with q levels U-type designs, denoted as $\mathcal{U}(n; q^s)$. A design d in $\mathcal{U}(n; q^s)$ can be presented as an $n \times s$ matrix with entries $1, 2, \ldots, q$, with each element occurring equally often in each column. From Definition 1, it is obvious that the Sudoku design X with order p is a member of $\mathcal{U}(p; p^p)$.

Definition 2. Let $C = (c_1, c_2, \ldots, c_p)$ be any random permutation of $\{1, 2, \ldots, p\}$, and k is an integer between 0 and p, then $T(C, k) = (c_{k+1}, c_{k+2}, \ldots, c_p, c_1, \ldots, c_k)$ is called as k-step right shift operator.

For a design $d \in \mathcal{U}(n; q^s)$ or equivalently for any $U(n; q^s)$, the measures of uniformity, discrete discrepancy, denoted as $DD(d)$, can be expressed in the following closed form

$$[DD(d; a, b)]^2 = -\left(\frac{a + (q-1)b}{q}\right)^s + \frac{a^s}{n} + \frac{2b^s}{n^2}\sum_{i=1}^{n}\sum_{j=i+1}^{n}\left(\frac{a}{b}\right)^{\psi_{ij}}, \tag{1}$$

where $a > b > 0$, and ψ_{ij} is the meeting number between the ith and the jth rows of d, i.e., the number of elements for which the ith and the jth rows of d take the same value. We can refer Qin and Fang (2004) for details.

According to Qin and Fang (2004), we have the attained lower bound of $[DD(d; a, b)]^2$, which is given in the following lemma. This lemma will help us to obtain uniformity of the constructed Sudoku designs.

Lemma 1. *Let $d \in \mathcal{U}(n; q^s)$ be U-type. Then*

$$[DD(d; a, b)]^2 \geq LDD(d; a, b), \tag{2}$$

where

$$LDD(d; a, b) = -\left(\frac{a + (q-1)b}{q}\right)^s + \frac{a^s}{n} + \frac{(n-1)[b(\gamma + 1 - \psi) + a(\psi - \gamma)]b^s}{nb}\left(\frac{a}{b}\right)^{\gamma},$$