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On hitting times for a simple random walk on dense Erdös–Rényi random graphs

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ABSTRACT

Let $(V, \mathcal{E})$ be a realization of the Erdös–Rényi random graph model $G(N, p)$ and $(X_n)_{n\in\mathbb{N}}$ be a simple random walk on it. We study the size of $\sum_{i\in V} \pi_i h_{ij}$ where $\pi_i = d_i/2|\mathcal{E}|$ for d_i the number of neighbors of node i and h_{ij} the hitting time for $(X_n)_{n\in\mathbb{N}}$ between nodes i and j. We always consider a regime of $p = p(N)$ such that realizations of $G(N, p)$ are almost surely connected as $N \to \infty$. Our main result is that $\sum_{i\in V} \pi_i h_{ij}$ is almost surely of order $N(1 + o(1))$ as $N \to \infty$. This coincides with previous non-rigorous results in the physics literature (Sood et al., 2004). Our techniques are based on large deviation bounds on the number of neighbors of a typical node and the number of edges in $G(N, p)$ (Chung and Lu, 2006) together with bounds on the spectrum of the (random) adjacency matrix of $G(N, p)$ (Erdős et al., 2011).

1. Introduction

Random walks have been used since a couple of years to investigate properties of finite and infinite graphs, e.g., Doyle and Snell (1984), Lovász (1993), Woess (2000), partially as part of a larger program for developing a theory of probability on finite and infinite graphs that accounts for their intrinsic geometrical structure, e.g., Aldous and Fill (in preparation), Grimmett (2010), Levin et al. (2009), Lyons and Peres (in preparation), and partially as an independent and exciting research area with its own rights. Meanwhile, the study of random graph models has received great attention not only within the probability community, e.g., Bollobás (2001), Chung and Lu (2006), Durrett (2006), Janson et al. (2000), Kolchin (1999), Van Der Hofstad (2009), but also within the physics, biology, engineering, computer sciences and social sciences communities, among others. In this paper, we give a small contribution to that program by studying some properties of hitting times of a random walk on Erdös–Rényi random graphs.

Let $\mathbb{G} := (\mathcal{G}, \mathcal{F}, \mathbb{P}_p)$ denote the probability space of the Erdös–Rényi random graph model $G(N, p)$ on N vertices. More precisely, $\mathcal{G}$ is the set of all graphs on N vertices, $\mathcal{F}$ is its power set, and $\mathbb{P}_p$ is the probability measure for which every edge is created, independently one from each other, with probability $p \in [0, 1]$. We will call a realization of such a graph $(V, \mathcal{E})$. We say that an event $A_N \subset \mathcal{G}$ happens asymptotically almost surely (abbreviated by a.a.s.) if $\mathbb{P}_p(A_N) \to 1$ as $N \to \infty$. Given $(V, \mathcal{E}) \in \mathcal{G}$, let $(X_n)_{n\in\mathbb{N}}$ be a simple random walk on $(V, \mathcal{E})$, i.e., $(X_n)_{n\in\mathbb{N}}$ is the discrete time Markov Chain with state space V

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and transition probabilities given by

$$p_{ij}^n := \mathbb{P}(X_{n+1} = j \mid X_n = i) = \begin{cases} 1/d_i & \text{if } ij \in \mathcal{E} \\ 0 & \text{otherwise.} \end{cases}$$

Here d_i is the degree of vertex $i \in V$. If $(V, \mathcal{E})$ is connected, it is well known that:

- $(X_n)_{n\in\mathbb{N}}$ has a unique stationary distribution defined by the vector
$$\pi = (\pi_1, \dots, \pi_N)^T \quad \text{where } \pi_i := d_i/(2|\mathcal{E}|).$$
- For $i, j \in V$, h_{ij} the expected number of steps $(X_n)_{n\in\mathbb{N}}$ takes to visit j when starting from i is (a.s.) finite.

The quantity h_{ij} is called *hitting (or access) time for* $(X_n)_{n\in\mathbb{N}}$ *between i and j*; see (2) below. Note that h_{ij} is a random variable on $\mathbb{G}$ and is itself an expectation with respect to the law of $(X_n)_{n\in\mathbb{N}}$ as well. In the present note, we would like to estimate the so-called *random target time*, defined by

$$H_j := \sum_{i\in V} \pi_i h_{ij}, \tag{1}$$

for the certain regime of $p = p(N)$ such that $(V, \mathcal{E})$ is a.a.s. connected. In Levin et al. (2009, Chapter 10) connections between H_j and some other relevant quantities (e.g., mixing time for random walks) are shown. Our main motivation is to give a rigorous proof of $H_j = N + o(N)$ for dense Erdös–Rényi graphs, as proposed in the physics work of Sood et al. (2004). In principle, $(V, \mathcal{E})$ is a.a.s. connected for $p = p_c := \kappa \frac{\log N}{N}$ for $\kappa > 1$ constant (cf. Durrett, 2006, Section 2.8 and references therein). In this note, it will be necessary to take $p(N) = \Omega((\log N)^{2C\xi-1})$ (therefore much larger than p_c), because we use results on the spectrum of A only available in this regime (Erdős et al., 2011) (cf. proof of Proposition 3.2 in our Section 3). However, we conjecture that our results are true already for $p \geq p_c$.

The rest of this note is organized as follows. In Section 2 we give the definitions and our main results on the order of magnitude of H_j, together with some applications estimating the order of magnitude of the so-called *random starting time*, see (6), and commenting on the order of magnitude of the commute time for $(X_n)_{n\in\mathbb{N}}$; see (8). Section 3 explains how to take advantage of a spectral decomposition for h_{ij}, for which every term will be bounded depending on the regime of $p = p(N)$ and finally the a.a.s. order of magnitude will be determined. The bounds on the degree of nodes and on the number of edges are valid for every regime of $p = p(N)$, though they are sharper in the regime where $(V, \mathcal{E})$ becomes a.a.s. connected (and therefore also in the regime where our main theorem is stated for). Our technique could eventually be applied to the largest component in other regimes of $G(N, p)$, provided corresponding bounds on the spectrum of its (random) adjacency matrix.

2. Definitions and main results

Let $(V, \mathcal{E})$ be an a.a.s. connected random graph in $\mathbb{G}$ and $(X_n)_{n\in\mathbb{N}}$ be a simple random walk taking values on V as defined above. Its law is denoted by $\mathbb{P}$ and the corresponding expectation by $\mathbb{E}$, and as usual, for $i \in V$, let $\mathbb{P}_i(\cdot) = \mathbb{P}(\cdot|X_0 = i)$ and $\mathbb{E}_i(\cdot) = \mathbb{E}(\cdot|X_0 = i)$. As mentioned in Introduction, our aim is to estimate the order of magnitude of quantities involving hitting times of $(X_n)_{n\in\mathbb{N}}$. Let τ_j be the first time $(X_n)_{n\in\mathbb{N}}$ is at $j \in V$, i.e.

$$\tau_j := \inf\{m > 0 : X_m = j\}. \tag{2}$$

The hitting time h_{ij} is then given by $h_{ij} := \mathbb{E}_i(\tau_j)$. We will represent h_{ij} in terms of eigenvalues and eigenvectors of the adjacency matrix of $(V, \mathcal{E})$. Let D be the diagonal matrix with entries $(D)_{ii} = 1/d_i$ and A be the adjacency matrix of $(V, \mathcal{E})$. Consider the symmetric (random) matrix

$$B := D^{1/2}AD^{1/2}$$

and its (random) eigenvalues $1 = \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N$. The corresponding orthonormal (random) eigenvectors are denoted by $v_1, \dots, v_N$, and there components are $v_k = (v_{kj})_{j=1,\dots,N}$. Note that the positive (random) vector $w := (\sqrt{d_1}, \dots, \sqrt{d_N})$ satisfies $B \cdot w = 1 \cdot w$; therefore by the Perron–Frobenius theorem $v_{1j} = \sqrt{d_j/2|\mathcal{E}|}$. Here $|\mathcal{E}|$ is the number of edges in $(V, \mathcal{E})$. For later use, remark that:

- For every $k = 2, \dots, N$
$$0 = v_k \cdot v_1 = \sum_{j=1}^N v_{kj} v_{1j} = \frac{1}{\sqrt{2|\mathcal{E}|}} \sum_{j=1}^N v_{kj}\sqrt{d_j} \Rightarrow \sum_{j=1}^N v_{kj}\sqrt{d_j} = 0 \tag{3}$$
- The matrix $V := (v_{kj})_{k,j=1,\dots,N}$ is unitary, and hence its rows and its columns form an orthonormal set; therefore
$$1 = \sum_{j=1}^N v_{kj}^2 = \sum_{k=1}^N v_{kj}^2. \tag{4}$$