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A new proof of an Engelbert–Schmidt type zero–one law for time-homogeneous diffusions☆

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ABSTRACT

In this paper we give a new proof to an Engelbert–Schmidt type zero–one law for time-homogeneous diffusions, which provides deterministic criteria for the convergence of integral functional of diffusions. Our proof is based on a slightly stronger assumption than that in Mijatović and Urusov (2012a) and utilizes stochastic time change and Feller's test of explosions. It does not rely on advanced methods such as the first Ray–Knight theorem, William's theorem, Shepp's dichotomy result for Gaussian processes or Jeulin's lemma as in the previous literature (see Mijatović and Urusov (2012a) for a pointer to the literature). The new proof has an intuitive interpretation as we link the integral functional to the explosion time of an associated diffusion process.

1. Introduction

The Engelbert–Schmidt zero–one law was initially proved in the standard Brownian motion case (see Engelbert and Schmidt, 1981 or Proposition 3.6.27, p. 216 of Karatzas and Shreve, 1991). Engelbert and Tittel (2002) obtain a generalized Engelbert–Schmidt type zero–one law for the integral functional $\int_0^t f(X_s)ds$, where f is a non-negative Borel measurable function and X is a strong Markov continuous local martingale. In an expository paper, Mijatović and Urusov (2012a) consider the case of a one-dimensional time-homogeneous diffusion and their Theorem 2.11 gives the corresponding Engelbert–Schmidt type zero–one law for time-homogeneous diffusions. Their proof relies on their Lemma 4.1, in which they provide two proofs without using Jeulin's lemma (see Lemma 3.1 of Engelbert and Tittel, 2002). The first proof is based on William's theorem (Chapter VII, Corollary 4.6, p. 317, Revuz and Yor, 1999). The second proof is based on the first Ray–Knight theorem (Chapter XI, Theorem 2.2, p. 455, Revuz and Yor, 1999). In Khoshnevisan et al. (2006), a similar question on the convergence of integral functional of diffusions is treated under different assumptions with answers given in different terms (see the discussion on p. 2 of Mijatović and Urusov, 2012a). Here our discussion is based on the setting in Mijatović and Urusov (2012a).

The contribution of this paper is two-fold. First, under a slightly stronger assumption that f is positive, we complement the study of the Engelbert–Schmidt type zero–one law in Mijatović and Urusov (2012a) with a new simple proof that circumvents advanced tools such as William's theorem and the first Ray–Knight theorem, which are employed by them to prove their Lemma 4.1. As discussed on p. 10 in their paper, in the previous literature, this lemma has been proven using the first Ray–Knight theorem with either Shepp's dichotomy result (Shepp, 1966) for Gaussian processes (see Engelbert and Schmidt, 1987) or Jeulin's lemma (see Engelbert and Tittel, 2002). Second, through stochastic time change, we establish a link between the integral functional of diffusions and the explosion time of another associated time-homogeneous diffusion.

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Our proof has an intuitive interpretation: "*the convergence/divergence of integral functional of diffusions is equivalent to the explosion/non-explosion of the associated diffusion*". Mijatović and Urusov (2012a) give a proof (see their Proposition 2.12) of Feller's test of explosions as an application of their results, and Theorem 2.1 here provides a converse to their result under a slightly stronger assumption.

The paper is organized as follows. Section 2 describes the probabilistic setting, states results on stochastic time change and gives the new proof. Section 3 concludes the paper and provides future research directions.

2. Main result

Given a complete filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,\infty)}, P)$ with state space $J = (\ell, r)$, $-\infty \leqslant \ell < r \leqslant \infty$, and $\mathcal{F}$ is right continuous (i.e. $\mathcal{F}_t = \mathcal{F}_{t+}$ for $t \in [0, \infty)$). Assume that the J-valued diffusion $Y = (Y_t)_{\{t\in[0,\infty)\}}$ satisfies the stochastic differential equation (SDE)

$$dY_t = \mu(Y_t)dt + \sigma(Y_t)dW_t, \quad Y_0 = x_0, \tag{1}$$

where W is a $\mathcal{F}_t$-Brownian motion and $\mu, \sigma : J \to \mathbb{R}$ are Borel measurable functions satisfying the Engelbert–Schmidt conditions

$$\forall x \in J,\ \sigma(x) \neq 0, \quad \frac{1}{\sigma^2(\cdot)}, \ \frac{\mu(\cdot)}{\sigma^2(\cdot)} \in L^1_{loc}(J), \tag{2}$$

where $L^1_{loc}(J)$ denotes the class of locally integrable functions, i.e., the functions $J \to \mathbb{R}$ are integrable on compact subsets of J. This condition (2) guarantees that the SDE (1) has a unique in law, weak solution that possibly exits its state space J (see Theorem 5.15, p. 341, Karatzas and Shreve, 1991).

Denote the possible explosion time of Y from its state space by ζ, i.e. $\zeta = \inf\{u > 0, Y_u \notin J\}$, which means that P-a.s. on $\{\zeta = \infty\}$ the trajectories of Y do not exit J, and P-a.s. on $\{\zeta < \infty\}$, we have $\lim_{t\to\zeta} Y_t = r$ or $\lim_{t\to\zeta} Y_t = \ell$. Y is defined such that it stays at its exit point, which means that l and r are absorbing boundaries. The following terminology is used: Y exits the state space J at r means $P\left(\zeta < \infty, \lim_{t\to\zeta} Y_t = r\right) > 0$.

Let b be a Borel measurable function such that $\forall x \in J$, $b(x) \neq 0$, and assume the following local integrability condition

$$\frac{b^2(\cdot)}{\sigma^2(\cdot)} \in L^1_{loc}(J). \tag{3}$$

Define the function $\varphi_t := \int_0^t b^2(Y_u)du$, for $t \in [0, \zeta]$.

Lemma 2.1. *Assume (3) and also that $\forall x \in J$, $b(x) \neq 0$; then φ_t is a strictly increasing function for $t \in [0, \zeta]$ and it is absolutely continuous for a.a. ω's on compact intervals of $[0, \zeta)$. Furthermore, $\varphi_t < \infty$ for $t \in [0, \zeta)$.*

Proof. Since $\forall x \in J$, $b(x) \neq 0$, φ_t is a strictly increasing function for $t \in [0, \zeta]$. Note that φ_t, $t \in [0, \zeta)$ is represented as a time integral, and the continuity follows. It is a standard result that the condition (3) implies that $\varphi_t < \infty$ for $t \in [0, \zeta)$ (see the proof after (8) on p. 4 and p. 5 in Mijatović and Urusov, 2012b). □

The following result is about the stochastic time change.

Proposition 2.1 (*Theorem 3.2.1 of Cui, 2013*). *Recall that $\varphi_t = \int_0^t b^2(Y_u)du$, $t \in [0, \zeta]$. Assume that the conditions (2), (3) are satisfied, and $\forall x \in J$, $b(x) \neq 0$.*

(i) *Under $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,\infty)}, P)$, define*

$$T_t := \begin{cases} \inf\{u \geqslant 0 : \varphi_{u\wedge\zeta} > t\}, & \text{on } \left\{0 \leqslant t < \varphi_\zeta\right\}, \\ \infty, & \text{on } \left\{\varphi_\zeta \leqslant t < \infty\right\}. \end{cases} \tag{4}$$

Define a new filtration $\mathcal{G}_t := \mathcal{F}_{T_t}$, $t \in [0, \infty)$, and a new process $X_t := Y_{T_t}$, on $\left\{0 \leqslant t < \varphi_\zeta\right\}$. Then X_t is $\mathcal{G}_t$-adapted and we have the stochastic representation

$$Y_t = X_{\int_0^t b^2(Y_s)ds} = X_{\varphi_t}, \ P\text{-a.s.} \quad \text{on } \{0 \leqslant t < \zeta\}, \tag{5}$$

and the process X is a time-homogeneous diffusion, which solves the following SDE under P

$$dX_t = \frac{\mu(X_t)}{b^2(X_t)}\mathbb{1}_{t\in[0,\varphi_\zeta)}dt + \frac{\sigma(X_t)}{b(X_t)}\mathbb{1}_{t\in[0,\varphi_\zeta)}dB_t, \quad X_0 = x_0, \tag{6}$$

where B_t is the $\mathcal{G}_t$-adapted Dambis–Dubins–Schwartz Brownian motion under P defined in the proof.

(ii) *Define $\zeta^X := \inf\{u > 0 : X_u \notin J\}$, and then $\zeta^X = \varphi_\zeta = \int_0^\zeta b^2(Y_s)ds$, P-a.s., and we can rewrite the SDE (6) under P as*

$$dX_t = \frac{\mu(X_t)}{b^2(X_t)}\mathbb{1}_{t\in[0,\zeta^X)}dt + \frac{\sigma(X_t)}{b(X_t)}\mathbb{1}_{t\in[0,\zeta^X)}dB_t, \quad X_0 = x_0. \tag{7}$$

(iii) *The event $\left\{\limsup_{t\to\zeta} Y_t = r\right\}$ is equal to $\left\{\limsup_{t\to\zeta^X} X_t = r\right\}$. Similarly for the case of the left boundary ℓ, the case of* lim inf, *and* lim.